ON THE LOCAL PROPERTY OF $\overline{N}, p_n, \alpha_n; \delta_k$ SUMMABILITY OF A FACTORED FOURIER SERIES

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ABSTRACT

In this paper we have established a theorem on the local property of $\overline{N}, p_n, \alpha_n; \delta_k$ summability of a factored Fourier series.

Key Words: $\overline{N}, p_n$ - summability, $\overline{N}, p_n, \alpha_n$ - summability, $\overline{N}, p_n, \alpha_n; \delta_k$ - summability and Fourier series.

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1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real constants such that

\[ P_n = \sum_{v=0}^{n} p_v \rightarrow \infty \text{ as } n \rightarrow \infty \quad (P_{-i} = p_{-i} = 0, \forall i \geq 1) \]

The sequence -to-sequence transformation

\[ t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \]

defines the sequence $\{t_n\}$ of the $\overline{N}, p_n$-means of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable $\overline{N}, p_n$-summable $\overline{N}, p_n$-summability and $\overline{N}, p_n, \alpha_n; \delta_k$-summability if

\[ \sum_{n=1}^{\infty} \left( \frac{P_n}{P_{n-1}} \right)^{k-1} |t_n - t_{n-1}|^k < \infty. \]

For $k=1$, $\overline{N}, p_n$-summability is same as $\overline{N}, p_n$-summability.

When $p_n = 1$ for all $n$ and $k = 1$, $\overline{N}, p_n$-summability is same as $|C,1|$-summability.

Also if we take $k = 1$ and $p_n = \frac{1}{(n+1)}$, summability is equivalent to the summability $|R, \log n, 1|$.  

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For any sequence \( \{c_n\} \) we use the following notation
\[
\Delta c_n = c_n - c_{n-1}, \quad \Delta^2 c_n = \Delta(\Delta c_n).
\]
A sequence \( \{\lambda_n\} \) is said to be convex if \( \Delta^2 \lambda_n \geq 0 \) for every positive integer ‘n’.

Let \( \{\alpha_n\} \) be any sequence of positive numbers. The series \( \sum a_n \) is said to be \( p_n, \alpha_n \) summable \( [N, p_n, \alpha_n] \), \( k \geq 1 \), if
\[
\sum_{n=1}^{\infty} \alpha_n^{k-1} |t_n - t_{n-1}| < \infty,
\]
where \( \{t_n\} \) is as defined in (1.2).

Let \( \{\alpha_n\} \) be any sequence of positive numbers. The series \( \sum a_n \) is said to be \( p_n, \alpha_n, \delta \) summable if
\[
\sum_{n=1}^{\infty} \alpha_n^{d+k-1} |t_n - t_{n-1}|^k < \infty.
\]

Let \( f(t) \) be a periodic function with period \( 2\pi \), integrable in the sense of Lebesgue over \((-\pi, \pi)\). Without loss of generality we may assume that the constant term in the Fourier series of \( f(t) \) is zero, so that
\[
f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).
\]

2. KNOWN THEOREMS

Dealing with the \( [N, p_n] \) -summability of an infinite series Bor[1] proved the following theorem:

2.1. THEOREM:

Let \( k \geq 1 \) and let the sequences \( \{p_n\} \) and \( \{\lambda_n\} \) be such that
\[
(2.1.1) \quad \Delta X_n = O\left(\frac{1}{n}\right),
\]
\[
(2.1.2) \quad \sum_{n=1}^{\infty} X_n^{k-1} \frac{|\lambda_n|^k + |\lambda_{n+1}|^k}{n} < \infty,
\]
\[
(2.1.3) \quad \sum_{n=1}^{\infty} (X_n^k + 1) |\Delta \lambda_n| < \infty,
\]
where \( X_n = \frac{P_n}{np_n} \). Then the summability \( [N, p_n] \) of the series \( \sum_{n=1}^{\infty} A_n(t) \lambda_n X_n \) at a point can be ensured by the local property.
2.2. REMARK:

It is known that if $\{\lambda_n\}$ is a convex sequence and $\sum n^{-1} \lambda_n$ is convergent, then $\lambda_n \geq \lambda_{n+1} \geq 0$, $\lambda_n \log n = O(1)$ and $\sum \log n \Delta \lambda_n < \infty$.

In the present paper, we have proved a theorem on the local property of $\vert \bar{N}, p_n, \alpha_n; \delta \vert_k$-summability of a factored Fourier series.

3. MAIN THEOREM

Let $k \geq 1$. Suppose $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1} \lambda_n$ is convergent. Let $\{\alpha_n\}$ and $\{p_n\}$ be a sequence of positive numbers such that

\begin{equation}
\Delta X_n = O \left( \frac{1}{n} \right),
\end{equation}

\begin{equation}
\sum_{n=m+1}^{n} \alpha_n^{\delta+k-1} \left( p_n / p_{n-1} \right)^k \left( \frac{1}{p_n} \right) = O \left( \frac{1}{p_n} \right),
\end{equation}

\begin{equation}
\sum_{n=1}^{\infty} X_n^{k-1} \frac{1}{n} \left| \lambda_n \right|^k < \infty,
\end{equation}

\begin{equation}
\sum_{n=1}^{\infty} (X_n^k + 1) \left| \Delta \lambda_n \right| < \infty,
\end{equation}

and

\begin{equation}
\sum_{n=2}^{\infty} \alpha_n^{\delta+k-1} \frac{1}{n^k} \left| \lambda_n \right|^k < \infty,
\end{equation}

where $X_n = \frac{P_n}{np_n}$. Then the summability $\vert \bar{N}, p_n, \alpha_n; \delta \vert_k, k \geq 1, \delta \geq 0$ of the series $\sum_{n=1}^{\infty} A_n(t) \lambda_n X_n$ at a point can be ensured by the local property.

In order to prove the above theorem we require the following lemma:

4. LEMMA

Let $k \geq 1$. Suppose $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1} \lambda_n$ is convergent and $\{p_n\}$ be a sequence such that the conditions (3.1)-(3.5) are satisfied. If $\{s_n\}$ is bounded then the series $\sum_{n=1}^{\infty} a_n \lambda_n X_n$ is $\vert \bar{N}, p_n, \alpha_n; \delta \vert_k, k \geq 1, \delta \geq 0$-summable when $\{\alpha_n\}$ is any sequence of positive numbers.

5. PROOF OF THE LEMMA

Let $\{T_n\}$ denote the $\vert \bar{N}, p_n \vert$-mean of the series $\sum_{n=1}^{\infty} a_n \lambda_n X_n$. Then by definition we have
\[ T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{\nu} \alpha_r \lambda_r X_r \]

\[ = \frac{1}{P_n} \sum_{v=0}^{n} (P_v - P_{v-1}) \alpha_v \lambda_v X_v , X_0 = 0. \]

For \( n \geq 1 \), we have

\[ T_n - T_{n-1} = \frac{P_n}{P_{n-1} P_n} \sum_{v=1}^{n} P_{v-1} \alpha_v \lambda_v X_v. \]

\[ T_n - T_{n-1} = -\frac{P_n}{P_{n-1} P_n} \sum_{v=1}^{n-1} P_v s_v \lambda_v X_v + \frac{P_n}{P_{n-1} P_n} \sum_{v=1}^{n-1} P_v s_v X_v \Delta \lambda_v \]

So,

\[ + \frac{P_n}{P_{n-1} P_n} \sum_{v=1}^{n-1} P_v s_v \lambda_{v+1} \Delta X_v + \frac{P_n s_n \lambda_n X_n}{P_n}. \]

(by Abel’s transformation)

\[ = T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \quad \text{(say)} \]

To complete the proof of the Lemma using Minokowski’s inequality, it is sufficient to show that

\[ \sum_{n=1}^{\infty} \alpha_n^{\delta k+1} |T_{n,i}|^k < \infty \quad \text{for } i = 1,2,3,4. \]

Now, we have

\[ \sum_{n=2}^{m+1} \alpha_n^{\delta k+1} |T_{n,i}|^k \]

\[ = \sum_{n=2}^{m+1} \alpha_n^{\delta k+1} \left| \frac{P_n}{P_{n-1} P_n} \sum_{v=1}^{n-1} p_v s_v \lambda_v X_v \right|^k \]

\[ \leq \sum_{n=2}^{m+1} \alpha_n^{\delta k+1} \left( \frac{P_n}{P_{n-1} P_n} \right) \left( \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v |\lambda_v|^k |s_v|^k |X_v|^k \right) \left( \frac{1}{P_{n-1} P_n} \right) \]

\[ = O(1) \sum_{v=1}^{m} |\lambda_v|^k |s_v|^k |X_v|^k \sum_{n=\nu+1}^{m+1} \alpha_n^{\delta k+1} \left( \frac{P_n}{P_{n-1} P_n} \right) \left( \frac{1}{P_{n-1}} \right) \]

\[ = O(1) \sum_{v=1}^{m} |\lambda_v|^k |X_v|^k \frac{P_n}{P_v} , \text{ by (3.2)} \]

\[ = O(1) \sum_{v=1}^{m} |\lambda_v|^k |X_v|^k \frac{P_n}{P_v} P_v \frac{P_v}{v} , \text{ as } X_n = \frac{P_n}{np_n} \]

\[ = O(1) \sum_{v=1}^{m} |X_v|^k \lambda_v^k \frac{1}{v} \]

\[ = O(1) \quad \text{as } m \to \infty , \text{ by (3.3)} . \]

Again,
\[
\sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} |T_{n,2}|^k \\
= \sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \left| \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v X_v \Delta \lambda_v \right|^k \\
\leq \sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \left( \frac{p_n}{p_n p_{n-1}} \right)^k \left( \frac{1}{p_{n-1}} \right)^k \left( \sum_{v=1}^{n-1} P_v |\Delta \lambda_v| \right) \left| X_v \right|^k \left( \frac{1}{P_{n-1}} \right)^k \sum_{v=1}^{n-1} P_v |\Delta \lambda_v| \right)^{k-1}
\]

Since
\[
\sum_{r=1}^{n-1} P_v |\Delta \lambda_v| \leq P_{n-1} \sum_{v=1}^{m+1} |\Delta \lambda_v| \Rightarrow \frac{1}{p_{n-1}} \sum_{r=1}^{n-1} P_v |\Delta \lambda_v| \leq \sum_{v=1}^{n-1} P_v |\Delta \lambda_v| = O(1)
\]

\[
= O(1) \sum_{v=1}^{m} P_v |\Delta \lambda_v| X_v^k \sum_{n=v+1}^{m+1} \alpha_n^{\delta k+k-1} \left( \frac{p_n}{p_n p_{n-1}} \right)^k \left( \frac{1}{p_{n-1}} \right)^k
\]

\[
= O(1) \sum_{v=1}^{m} |\Delta \lambda_v| X_v^k \text{, by (3.2)}
\]

\[
= O(1) \text{ as } m \to \infty \text{, by (3.4)}.
\]

Further,
\[
\sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} |T_{n,3}|^k \\
= \sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \left| \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta \lambda_{v+1} \Delta X_v \right|^k \\
= O(1) \sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \left( \frac{p_n}{p_n p_{n-1}} \right)^k \left( \sum_{v=1}^{n-1} P_v |\Delta \lambda_{v+1}| \right)^k \left( \frac{1}{p_{n-1}} \right)^k \sum_{v=1}^{n-1} P_v |\Delta \lambda_{v+1}| \right)^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} P_v |\lambda_{v+1}| X_v^k \sum_{n=v+1}^{m+1} \alpha_n^{\delta k+k-1} \left( \frac{p_n}{p_n p_{n-1}} \right)^k \left( \frac{1}{p_{n-1}} \right)^k \sum_{v=1}^{n-1} P_v |\lambda_{v+1}| \right)^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} P_v |\lambda_{v+1}| X_v^k \text{, by (3.2)}
\]

\[
= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| X_v^k \frac{p_v}{p_v} \frac{p_v}{p_v}, \text{ as } X_n = \frac{p_n}{np_n}
\]

\[
= O(1) \sum_{v=1}^{m} X_v^k \left| \frac{\lambda_{v+1}}{v} \right|^k
\]

\[
= O(1) \text{ as } m \to \infty \text{, by (3.3)}.
\]
Now,

\[
\sum_{n=2}^{m+1} \alpha_n^{\delta_{k-1}} |T_{n,A}|^k
\]

\[
= \sum_{n=2}^{m+1} \alpha_n^{\delta_{k-1}} \left| \frac{p_n s_n \lambda_n X_n}{p_n} \right|^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \alpha_n^{\delta_{k-1}} X_n^k |\lambda_n|^k \left( \frac{p_n}{p_n} \right)^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \alpha_n^{\delta_{k-1}} \frac{\lambda_n^k}{n^k}, \text{ as } X_n = \frac{p_n}{np_n}
\]

\[
= O(1) \quad \text{as } m \to \infty, \text{ by (3.5).}
\]

This completes the proof of the Lemma.

6. PROOF OF THE THEOREM

Since the behavior of the Fourier series, as far as convergence is concerned, for a particular value of \(x\) depends on the behavior of the function in the immediate neighborhood of this point only, the truth of the theorem is necessary consequence of the Lemma.

REFERENCES