COROTATIONAL METHOD AND DIFFERENTIAL GEOMETRY

Mohammed Khalil Ferradi
SETEC-TPI, 42/52 Quai de la Rapée 75012, Paris, France
E-mail: mohammed-khalil.ferradi@tpi.setec.fr

ABSTRACT
It's known that in nonlinear analysis of a 3D beam with the corotational method, we obtain a non-symmetric tangent stiffness matrix, even in the case of a conservative loading, this is due to the fact that the rotation in any point can no longer be described by a vector, as in the linear case, but by an orthogonal rotation matrix, that is an element of the special orthogonal group SO(3), which makes the configuration space of the beam to be non-Euclidean. We will try to prove that by replacing the directional derivative in the derivation of the tangent stiffness, by the covariant derivative, we will always obtain a symmetric matrix, even away from a non-equilibrium configuration.

Keywords : *3D beam element, corotational method, nonlinear analysis, differential geometry*.

1. INTRODUCTION
The main idea of the corotational method in the study of nonlinear element, is to separate the purely deformational motion from the rigid body motion, thus we can separate the computation of the local stiffness of the beam, from the corotational procedure, where we introduce all the geometric non linearity's, the procedure can be then applied to any two node element with twelve degree of freedom. It's one of the advantages of the method, it allow us to use linear higher order beam elements that we already have in our finite element library. The local behaviorwill bedescribed by local displacements and rotations, expressed in a local *moving* frame attached to the beam, in function of the global displacements and rotations that are expressed in the global frame. The choice of the moving frame attached to the beam, can have some incident on the convergence rate of the solution, different choice exist in the literature, for simplicity we will use in this paper the frame attached and centered at one of the beam's node as the element frame.

In the finite element formulation of a non-linear beam element, we will need to perform a linearization of the virtual work, this is done with the aim of a directional derivative, leading to the derivation of a non-symmetric tangent stiffness matrix, even for conservative loading.This is due to the non-linear structure of the configuration space, which represents the rotation at a beam's node with an orthogonal matrix. In structural analysis, we need to hold the symmetry of the problem for conservative loading, this has motivated the development of alternative formulations that use additive rotation vector to derive a symmetric stiffness matrix.
J.C. Simo [1] has shown for a geometrically exact beam model, that by replacing the directional derivative by a covariant derivative in the linearization process of the virtual work , we will always obtain a symmetric tangent stiffness matrix, even away from an equilibrium state, and this symmetric matrix correspond to the symmetric part of the non-symmetric stiffness matrix obtained with a directional derivative. This gives a strong justification for the symmetrizing process used to derive a symmetric tangent stiffness matrix.In general way, we can prove, as it's shown in Zefran& Kumar [4], that by using a symmetric connection in a Riemannian manifold, we will always obtain a symmetric tangent stiffness matrix.
In the corotational method we also use a symmetrizing process to derive a symmetric matrix. Crisfield propose to check the numerical results to verify that the quadratic convergence in a Newton-Raphson method is not impaired. In this paper we will follow the work of J.C. Simo [1], byreplacing the directional derivative by a covariant derivative in the corotational formulation, the matrix obtained will be then symmetric but unlike the geometrically exact method, this matrix do not exactly correspond to the symmetric part of the non-symmetric matrix obtained by the classical formulation, some terms will be missing.

2. CONFIGURATION SPACE AND RIEMANNIAN METRIC:
In the corotational method, we work in the discretized form of the beam into two nodes.The configuration space will be described by the position vector and the rotation matrix at each end nodes of the beam:

$$Q = \{\Phi = (\varphi_A, R_A, \varphi_B, R_B) \,/\, (\varphi_A, \varphi_B) \in \mathbb{R}^3 \times \mathbb{R}^3 \,, (R_A, R_B) \in \, SO(3) \times SO(3)\} \qquad 1.$$

For what follows we will introduce somenotations. For$v^T = (v_x \quad v_y \quad v_z) \in \mathbb{R}^3$,$w \in \mathbb{R}^3$and $R \in SO(3)$ we will have:

$$v \times w = \hat{v}w \quad , \quad \hat{v}R = v_R \text{with} \hat{v} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix} \in so(3)$$

Where so(3) is the space of skew symmetric matrices, and $\times$denote the vectorial product.
The tangent space to Q at a given configuration Φ, is obtained by superposing infinitesimal displacement and rotation at each node:

$$T_\Phi Q = \{V_\Phi = (v_1, \widehat{v_2}R_A, v_3, \widehat{v_4}R_B) \,/\, (v_1, v_3) \in \mathbb{R}^3 \times \mathbb{R}^3 \,, (\widehat{v_2}, \widehat{v_4}) \in \; so(3) \times so(3)\} \qquad 2.$$

The matrices $\widehat{v_2}R_A$ and $\hat{v}_4 R_B$ represents infinitesimal rotations superposed to R_A and R_B.
We will need to endowour configuration space Q with a Riemannian metric, for arbitrary
$U_\Phi \,, V_\Phi \in T_\Phi Q$we have:

$$\langle U_\Phi \,, V_\Phi \rangle = u_1 \cdot v_1 + u_3 \cdot v_3 + \frac{1}{2} tr((\widehat{u_2}R_A)^T \widehat{v_2}R_A) + \frac{1}{2} tr((\widehat{u_4}R_B)^T \widehat{v_4}R_B)$$

$$\langle U_\Phi \,, V_\Phi \rangle = u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3 + u_4 \cdot v_4 \qquad 3.$$

To obtain the equation 3 we have used the following relations:$tr(R^T D R) = tr(D) \;,\; tr(\hat{u}\hat{v}) = -2\, u \cdot v$
Where $\cdot$ denote the scalar product.
We will also need to define the Lie bracket:

$$[U_\Phi \,, V_\Phi] = U_\Phi V_\Phi - V_\Phi U_\Phi = \left(0 \,, (v_2 \times u_2)_{R_A}, 0 \,, (v_4 \times u_4)_{R_B}\right) \qquad 4.$$

With these additional structures that we have defined, we can determine a unique, torsion free, connection, associated to the Riemannian metric, and called the Levi-Civita connection.
We note that a torsion free (or symmetric)connectionis a connection ∇verifying:

$$\nabla_{U_\Phi} V_\Phi - \nabla_{V_\Phi} U_\Phi = [U_\Phi \,, V_\Phi] \qquad 5.$$

For arbitrary tangent vector field U_Φ, V_Φ, W_Φ, we have the following formula:

$$2\langle \nabla_{U_\Phi} V_\Phi \,, W_\Phi \rangle = \langle [U_\Phi \,, V_\Phi] \,, W_\Phi \rangle - \langle [U_\Phi \,, W_\Phi] \,, V_\Phi \rangle - \langle [V_\Phi \,, W_\Phi] \,, U_\Phi \rangle \qquad 6.$$

$$2\langle \nabla_{U_\Phi} V_\Phi \,, W_\Phi \rangle = (v_2 \times u_2) \cdot w_2 + (v_2 \times u_2) \cdot w_4 \qquad 7.$$

Thus, the Levi-Civita connection is expressedby :

$$\nabla_{U_\Phi} V_\Phi = \frac{1}{2} [U_\Phi \,, V_\Phi] \qquad 8.$$

From the expression of the connection we can see that$\nabla_{U_\Phi} U_\Phi = 0$, thus we can say that every curve whose tangent vector belongs to$T_\Phi Q$ is a geodesic.

3. THE CURVATURE

The Riemannian curvature is defined by:

$$R(U_\Phi \,, V_\Phi) W_\Phi = \nabla_{V_\Phi} \nabla_{U_\Phi} W_\Phi - \nabla_{U_\Phi} \nabla_{V_\Phi} W_\Phi + \nabla_{[U_\Phi \,, V_\Phi]} W_\Phi \qquad 9.$$

$$R(U_\Phi \,, V_\Phi) W_\Phi = \frac{1}{4} \left[V_\Phi \,, [U_\Phi \,, W_\Phi]\right] - \frac{1}{4} \left[U_\Phi \,, [V_\Phi \,, W_\Phi]\right] + \frac{1}{2} \left[[U_\Phi \,, V_\Phi], W_\Phi\right] \qquad 10.$$

From the Jacobi identity:

$$\left[[U_\Phi\ ,V_\Phi],W_\Phi\right] = \left[U_\Phi\ ,[V_\Phi\ ,W_\Phi]\right] - \left[V_\Phi\ ,[U_\Phi\ ,W_\Phi]\right] \qquad 11.$$

We obtain the expression of the Riemannian curvature:

$$R(U_\Phi\ ,V_\Phi)W_\Phi = \frac{1}{4}\left[[U_\Phi\ ,V_\Phi],W_\Phi\right] \qquad 12.$$

$$R(U_\Phi\ ,V_\Phi)W_\Phi = \frac{1}{4}\Big(0\,,\big(w_2\times(v_2\times u_2)\big)_{R_A},0\,,\big(w_4\times(v_4\times u_4)\big)_{R_B}\Big) \qquad 13.$$

The non-vanishing Riemannian curvature tensor shows that the true nature of the configuration space is non-Euclidean.

4. COROTATIONAL FORMULATION

In the corotational method, we need to define a*moving* frame that is attached to the beam. For simplicity, this frame will be taken as the triad attached to the node A, defined by the rotation matrix R_A. We will express the position vectors and the rotation matrices of the beam's nodes in this local frame, to form a local configuration$\Phi_l = (\varphi_{lA}, R_{lA}, \varphi_{lB}, R_{lB})$:

$$\varphi_{lA} = 0 \quad , \quad R_{lA} = R_A^T R_A = I \quad , \quad \varphi_{lB} = R_A^T\varphi_{BA} - \begin{Bmatrix} L_0 \\ 0 \\ 0 \end{Bmatrix} \quad , \quad R_{lB} = R_A^T R_B \qquad 14.$$

Where the subscript l design a local configuration,$L_0$the initial length of the beam and$\varphi_{BA} = \varphi_B - \varphi_A$.
The derivative of $\Phi_l$in the direction of the tangent vector$U_\Phi \in T_\Phi Q$is given by :

$$U_\Phi[\varphi_{lA}] = 0 \ , \ U_\Phi[R_{lA}] = 0 \ , \ U_\Phi[\varphi_{lB}] = R_A^T(u_{31} + \varphi_{BA}\times u_2)$$

$$U_\Phi[R_{lB}] = R_A^T\widehat{u_{42}}R_B = R_A^T\widehat{u_{42}}R_A R_A^T R_B = \widehat{R_A^T u_{42}}R_{lB} \qquad 15.$$

Where$u_{ij} = u_i - u_j$.

We can write:

$$U_\Phi[\Phi_l] = \Big(0\,,0\,,R_A^T(u_{31} + \varphi_{BA}\times u_2)\,,\widehat{R_A^T u_{42}}R_{lB}\Big) \qquad 16.$$

We will also need the second derivative of Φ_l in the direction of the tangent vector
$V_\Phi = (v_1, \widehat{v_2}R_A, v_3, \widehat{v_4}R_B)$:

$$V_\Phi U_\Phi[\varphi_{lB}] = R_A^T\big(u_{31}\times v_2 + v_{31}\times u_2 - v_2\times(\varphi_{BA}\times u_2)\big)$$

$$V_\Phi U_\Phi[R_{lB}] = R_A^T(\widehat{u_{42}}\widehat{v_4} - \widehat{v_2}\widehat{u_{42}})R_B = R_A^T(\widehat{u_{42}\times v_2} + \widehat{u_{42}}\widehat{v_{42}})R_B \qquad 17.$$

The tangent vector U_Φ of a configuration Φ is characterized by the vectors u_1 and u_3, corresponding to a displacements of the beam nodes, and by the vectors u_2 and u_4, that can be seen as vectors of an infinitesimal rotation of the beam's nodes. Weexpress u_l the *local* infinitesimal displacements and rotations of the beam's nodes, in the *moving* frame attached to the beam that we have assumed to be centered at the node A and with axes defined by the triad R_A, thus we have:

$$u_{l1} = 0 \quad , \quad u_{l2} = 0 \quad , \quad u_{l3} = R_A^T(u_{31} + \widehat{\varphi_{BA}}u_2) \quad , \quad u_{l4} = R_A^T u_{42} \qquad 18.$$

$u_l^T = \{u_{l1}^T \quad u_{l2}^T \quad u_{l3}^T \quad u_{l4}^T\}$ Representsthe local displacement and rotation vectors of the node A and B, expressed in the local moving frame.
We can express now the matrix F connecting the infinitesimal, global and local, variables:

$$u_l = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -R_A^T & R_A^T \hat{\varphi}_{BA} & R_A^T & 0 \\ 0 & -R_A^T & 0 & R_A^T \end{bmatrix}}_{F} u \qquad 19.$$

5. THE VIRTUAL WORK AND THE TANGENT STIFFNESS MATRIX

The virtual work, expressed in the local moving frame, is:

$$\delta W = (q_{il} - q_{el}) \cdot u_l \qquad 20.$$

Where $q_{il}^T = \{N_A^T \quad M_A^T \quad N_B^T \quad M_B^T\}$and $q_{el}^T = \{P_A^T \quad T_A^T \quad P_B^T \quad T_B^T\}$ represents the internal and external local generalized efforts, N_A and P_A the internal and external force vectors at node A, M_A and T_A the internal and external moment vectors at node A, and u_l is generally called the virtual displacement vector.
We note that at an equilibrium configuration we have$q_{il} - q_{el} = 0 \iff \delta W = 0$
To develop the expression of the virtual work, we make use of the relation $u_l = Fu$ to obtain:

$$\delta W = (q_{il} - q_{el}) \cdot Fu$$

$$\delta W = (N_B - P_B) \cdot R_A^T (u_{31} + \widehat{\varphi_{BA}} u_2) + (M_B - T_B) \cdot R_A^T u_{42} \qquad 21.$$

From the last expression, we can see easily that we have the following relation:

$$\delta W = U_\Phi[W] = \langle Q_{il} - Q_{el} \, , U_\Phi[\Phi_l] \rangle \qquad 22.$$

Where $Q_{il} = \left(N_A \, , \widehat{M_A} R_{lA} \, , N_B \, , \widehat{M_B} R_{lB}\right)andQ_{el} = \left(P_A \, , \widehat{T_A} R_{lA} \, , P_B \, , \widehat{T_B} R_{lB}\right)$.
$U_\Phi[W]$is the first derivative of the system energy W in the direction of U_Φ. The second derivative of W will be expressed by :

$$V_\Phi U_\Phi[W] = \langle V_\Phi[Q_{il} - Q_{el}] \, , U_\Phi[\Phi_l] \rangle + \langle Q_{il} - Q_{el} \, , V_\Phi U_\Phi[\Phi_l] \rangle \qquad 23.$$

We consider here only the case of conservative loading, then :

$$V_\Phi U_\Phi[W] = \underbrace{\langle V_\Phi[Q_{il}] \, , U_\Phi[\Phi_l] \rangle}_{material \;\; part} + \underbrace{\langle Q \, , V_\Phi U_\Phi[\Phi_l] \rangle}_{geometric \;\; part} \qquad 24.$$

Where we have maked use of the notation$Q = Q_{il} - Q_{el}$.
The main idea of the corotational method is to make use of the local tangent (symmetric)stiffness matrix K_l, that we may already have in our finite element library. This matrix is defined by :

$$\delta q_{il} = K_l v_l \qquad 25.$$

Thus :

$$\langle V_\Phi[Q_{il}] \, , U_\Phi[\Phi_l] \rangle = (K_l v_l) \cdot u_l = (K_m v) \cdot u \qquad 26.$$

Where$K_m = F^T K_l F$is a symmetric matrix, representing the material part of the tangent stiffness.

We calculate now the geometric part :

$$\langle Q\ , V_\Phi U_\Phi[\Phi_l]\rangle = (N_B - P_B)\cdot R_A^T\big(u_{31}\times v_2 + v_{31}\times u_2 - v_2\times(\varphi_{BA}\times u_2)\big) + \frac{1}{2}tr\left(\left(\left(\widehat{M_B}-\widehat{T_B}\right)R_{lB}\right)^T R_A^T(\widehat{u_{42}\times v_2}+\widehat{u_{42}}\widehat{v_{42}})R_B\right) \quad 27.$$

Knowing that :

$$tr\left(\left(\left(\widehat{M_B}-\widehat{T_B}\right)R_{lB}\right)^T R_A^T(\widehat{u_{42}\times v_2}+\widehat{u_{42}}\widehat{v_{42}})R_B\right) = -tr\left(R_A(\widehat{M_B-T_B})R_A^T(\widehat{u_{42}\times v_2}+\widehat{u_{42}}\widehat{v_{42}})\right) \quad 28.$$

And if we decompose the matrix $\widehat{u_{42}}\widehat{v_{42}}$ into a symmetric and a skew-symmetric part :

$$\widehat{u_{42}}\widehat{v_{42}} = \frac{1}{2}(\widehat{u_{42}}\widehat{v_{42}}+\widehat{v_{42}}\widehat{u_{42}}) + \frac{1}{2}(\widehat{u_{42}}\widehat{v_{42}}-\widehat{v_{42}}\widehat{u_{42}})$$
$$\widehat{u_{42}}\widehat{v_{42}} = \frac{1}{2}(\widehat{u_{42}}\widehat{v_{42}}+\widehat{v_{42}}\widehat{u_{42}}) + \frac{1}{2}(\widehat{u_{42}\times v_{42}}) \quad 29.$$

Wewill have :

$$\langle Q\ , V_\Phi U_\Phi[\Phi_l]\rangle = (N_B - P_B)\cdot R_A^T\big(u_{31}\times v_2 + v_{31}\times u_2 - v_2\times(\varphi_{BA}\times u_2)\big) + (M_B - T_B)\cdot R_A^T\left(u_{42}\times v_2 + \frac{1}{2}u_{42}\times v_{42}\right) \quad 30.$$

In the formula above, we have maked use of the relation $tr(AB) = 0$,with A and B are respectively, a symmetric and a skew-symmetric matrix.
If we compare the expression of geometric part obtained here with the one that we will have obtained by using a classical formulation of the corotational method, we will see that the term with $u_{42}\times v_{42}/2$ will be missing in the classical formulation.

The second derivativeof the system energy W will be given now by:

$$V_\Phi U_\Phi[W] = (K_m v)\cdot u + \left(K_g v\right)\cdot u \quad 31.$$

Where K_g is the geometric part of the tangent stiffness matrix.
We note that at an equilibrium configuration we have$q_{il} - q_{el} = 0$, thus K_g=0, this shows that we recover the symmetry of the tangent stiffness at an equilibrium configuration, as demonstrated in [2] for a geometrically exact beam model.
The expression of the second directional derivative of W can be written as the sum of a symmetric and a skew symmetric part :

$$V_\Phi[\delta W] = V_\Phi U_\Phi[W] = \left(K_{sym}\, v\right)\cdot u + (K_{ske}\, v)\cdot u \quad 32.$$

The skew symmetric part is definedby :

$$(K_{ske}\, v)\cdot u = \frac{1}{2}(V_\Phi U_\Phi - U_\Phi V_\Phi)[W] = \frac{1}{2}\langle Q\ , (V_\Phi U_\Phi - U_\Phi V_\Phi)[\Phi_l]\rangle \quad 33.$$

We will need to calculate $(V_\Phi U_\Phi - U_\Phi V_\Phi)[\Phi_l]$:

$$(V_\Phi U_\Phi - U_\Phi V_\Phi)[\varphi_{lB}] = R_A^T\big(u_2\times(\varphi_{BA}\times v_2) - v_2\times(\varphi_{BA}\times u_2)\big) = R_A^T\big(\varphi_{BA}\times(u_2\times v_2)\big) \quad 34.$$

$$(V_\Phi U_\Phi - U_\Phi V_\Phi)[R_{lB}] = R_A^T(\widehat{u_{42}}\widehat{v_4} - \widehat{v_2}\widehat{u_{42}} - \widehat{v_{42}}\widehat{u_4} + \widehat{u_2}\widehat{v_{42}})R_B = R_A^T(\widehat{u_4\times v_4} - \widehat{u_2\times v_2})R_B \quad 35.$$

$$= \widehat{R_A^T w} R_{lB} = (R_A^T w)_{R_{lB}}$$

Where$w = u_4 \times v_4 - u_2 \times v_2$
Thus :

$$\begin{aligned}(K_{ske}\, v) \cdot u &= \frac{1}{2}(N_B - P_B) \cdot R_A^T\big(\varphi_{BA} \times (u_2 \times v_2)\big) + \frac{1}{2}(M_B - T_B) \cdot R_A^T w \\ &= \langle Q\,, \nabla_{V_\Phi} U_\Phi[\Phi_l]\rangle \\ &= \nabla_{V_\Phi} U_\Phi[W]\end{aligned} \quad 36.$$

If we perform the covariant derivative of δWweobtain :

$$\begin{aligned}\langle \nabla_{V_\Phi} Q\,, U_\Phi[\Phi_l]\rangle &= V_\Phi[\langle Q\,, U_\Phi[\Phi_l]\rangle] - \langle Q\,, \nabla_{V_\Phi} U_\Phi[\Phi_l]\rangle \\ &= V_\Phi[\delta W] - \nabla_{V_\Phi} U_\Phi[W] \\ &= \big(K_{sym}\, v\big) \cdot u\end{aligned} \quad 37.$$

The tangent stiffness matrix obtained is symmetric, even for a configuration away from equilibrium, and corresponds to the symmetric part of the matrix obtained from the directional derivative of the virtual work.

6. CONCLUSION

In the classical formulation of the corotational method,the tangent stiffness matrix$K_t$is obtained by performing a directional derivative of the virtual work, instead of a covariant derivative. As it was pointed in [3], this matrix will be non-symmetric. Thisformulation is equivalent to take $\nabla_{V_\Phi} U_\Phi = 0$ in our calculation, which defines a non-symmetric connection, and as proven in [4], this will give arise to a non-symmetric tangent stiffness matrix. In the formulation proposed by Crisfield [3], the equilibrium equations are differentiated to obtain their linearized form.A non-symmetric geometric stiffness matrix is then derived, and as we have already shown, it will be different from the one obtained in this paper by performing a directional derivative of the virtual work, to obtain then the symmetric tangent stiffness matrix, we need to add to the classical formulation the missing terms and then symmetrize the result.

7. REFERENCES

[1]. J.C. Simo, The (symmetric) Hessian for geometrically nonlinear models in solid mechanics : Intrinsic definition and geometric interpretation, Comput. Methods Appl. Mech. Engrg. 96 (1992) 189-200.

[2]. J.C. Simo and L. Vu-Quoc, A three dimensional finite strain rod model. Part II :Computational aspects, Comput. Methods Appl. Mech. Engrg. 58 (1986) 79-116.

[3]. M.A. Criesfield, Non-Linear Finite Element Analysis of Solids and Structures, Volume 2 (John Wiley & Sons, 1997).

[4]. M. Zefran and R.V. Kumar, A geometric approach to the study of the cartesian stiffness matrix, Journal of Mechanical Design, 124 (2002) 30-38.

[5]. J. Oprea, Differential Geometry and Its Application, (Prentice Hall, 1997)

[6]. T. Frankel, The Geometry of Physics : An Introduction, Third edition (Cambridge University Press, 2011).