COMPUTATION OF THE EIGENVALUES AND EIGENFUNCTION OF GENERALIZED STURM-LIOUVILLE PROBLEMS VIA THE DIFFERENTIAL TRANSFORMATION METHOD

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ABSTRACT
This paper deals with the computation of the eigenvalues and eigenfunction of generalized Sturm-Liouville problems using the differential transformation method. A few numerical examples among will be presented to illustrate the merit of the method and comparison made with the regularized sampling method. These results show that the technique introduced here is accurate and easy to apply.


1. INTRODUCTION
The study of many physical phenomena, such as the vibration of strings, the interaction of atomic particles, or the earth’s free oscillations yields Sturm-Liouville (SL) eigenvalue problems. The general form of sturm-liouville problems that concerns this paper is

\[ \sum_{i=0}^{2} p_i(x)u^{(i)}(x) = \lambda q(x)u(x), \quad x \in J = [0, b] \]  \hspace{1cm} (1)

subject the boundary conditions

\[ \begin{bmatrix} a_{11}(\lambda) & a_{12}(\lambda) & b_{11}(\lambda) & b_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) & b_{21}(\lambda) & b_{22}(\lambda) \end{bmatrix} \begin{bmatrix} u(0) \\ u'(0) \\ u(b) \\ u'(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]  \hspace{1cm} (2)

where \( p_i(x), i = 0, 1, 2. \) is a real valued function satisfying \( p_i(x) \in L^2(0, b). \) We assume that \( u \) is sufficiently differentiable and that a unique solution of (1) exists. The values of \( \lambda \) for which the boundary value problem has nontrivial solution are called eigenvalues of (1). A nontrivial solution corresponding to an eigenvalue is called an eigenfunction.

The differential transform method (DTM) is a semi-numerical-analytic technique that formalizes the Taylor series in a totally different manner. The DTM was first introduced by Zhou in a study concerning electrical circuits [24]. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally time-consuming for large orders. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations. With this technique, the given partial differential equation and related initial conditions are transformed into a recurrence equation, that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. This method is useful for obtaining exact and approximate solutions of linear and nonlinear ordinary and partial differential equations. There is no need for linearization or perturbations, and large amounts of computational work and round-of errors are avoided. It has been used to solve effectively, easily and accurately a large class of linear and nonlinear problems with approximations. It is possible to solve a system of differential equations [7, 21, 22], two point boundary value problems [15], initial-value problems [17], differential algebraic equations [9], difference equations [3], differential difference equations [4], partial differential equations [18, 13, 16, 10], eigenvalue problems [14], fractional differential equations [20, 5], pantograph equations [19], one-dimensional Volterra integral and integro-differential equations [6, 23] and matrix differential equations [1] by using this method.

The aim of this paper is to extend the differential transformation method to solve equation (1). The accuracy of the numerical results will be compared with that of the regularized sampling method.
The paper is organized into five sections. The definitions and operations of differential transformation is introduced in Section 2. In Section 3, the differential transformation method is developed for the eigenvalues of Sturm-Liouville problems with quite general separated boundary conditions parameter. Some numerical examples are presented in Section 4. Finally, Section 5 provides conclusions of the study.

2. THE BASIC IDEA OF THE DIFFERENTIAL TRANSFORM METHOD

Differential transformation method has increasingly been recognized as powerful tools for attacking problems in applied physics and engineering. A general review of differential transformation is given in [2, 7, 13]. Hence, only properties important to the present goals are outlined in this section. Let

\[ Y(k) = \frac{1}{k!} \left[ \frac{d^k y(x)}{dx^k} \right]_{x=0} \]  

(3)

where \( y(x) \) is original function, and \( Y(k) \) is the transformed function which is called the T-function in brief. Differential inverse transformation of \( Y(k) \) is defined as follows

\[ y(x) = \sum_{k=0}^{\infty} x^k Y(k) \]  

(4)

in fact, from (3) and (4), we obtain

\[ y(x) = \sum_{k=0}^{\infty} x^k \left[ \frac{d^k y(x)}{k!} \right]_{x=0} \]  

(5)

Equation (5) implies that the concept of differential transformation is derived from the Taylor series expansion. In this study, we use lower-case letters to represent the original functions and upper-case letters to stand for the transformed functions (T-function).

Some fundamental mathematical operations performed by differential transformation are listed

\[
\begin{array}{ll}
\text{Original function} & \text{Transformed function} \\
\hline
u(x) = f(x) \pm z(x) & U(k) = F(k) \pm Z(k) \\
u(x) = \alpha f(x) & U(k) = \alpha F(k) \\
u(x) = \frac{d f(x)}{dx} & U(k) = (k+1)F(k+1) \\
u(x) = \frac{d^2 f(x)}{dx^2} & U(k) = (k+1)(k+2)F(k+2) \\
u(x) = \frac{d^m f(x)}{dx^m} & U(k) = (k+1)(k+2)\ldots(k+m)F(k+m) \\
u(x) = f(x)z(x) & U(k) = \sum_{r=0}^{k} F(r)Z(k-r) \\
u(x) = x^m, \quad m = 0, 1, \ldots & U(k) = \delta(k-m) = \begin{cases} 1 & \text{for} \quad k = m \\ 0 & k \neq m \end{cases}
\end{array}
\]

3. THE DESCRIPTION OF DIFFERENTIAL TRANSFORMATION SCHEME

Taking differential transformation of (1), we obtain

\[
\sum_{L=0}^{k} P_2(L)(k-L+1)(k-L+2)Y(k-L+2) + \sum_{L=0}^{k} P_1(L)(k-L+1)Y(k-L+1) + \sum_{L=0}^{k} P_0(L)Y(k-L) = \lambda \sum_{L=0}^{k} Q(L)Y(k-L)
\]

(6)
where the $P_i$ and $i = 0, 1, 2, \ldots, Y(k)$ and $Q(k)$ are T-functions of $p_1, y(x)$ and $q$ respectively.

\[
P_2(0)(k + 1)(k + 2)Y(k + 2) + \sum_{L=1}^{k} P_2(L)(k - L + 1)(k - L + 2)Y(k - L + 2) + \sum_{L=0}^{k} \left[ P_1(L)(k - L + 1)Y(k - L + 1) + P_0(L)Y(k - L) - \lambda Q(L)Y(k - L) \right] = 0
\]  

or

\[
Y(k + 2) = \frac{\sum_{L=0}^{k} P_2(L)(k - L + 1)(k - L + 2)Y(k - L + 2)}{P_2(0)(k + 1)(k + 2)} - \frac{\sum_{L=0}^{k} \left[ P_1(L)(k - L + 1)Y(k - L + 1) + P_0(L)Y(k - L) - \lambda Q(L)Y(k - L) \right]}{P_2(0)(k + 1)(k + 2)}
\]

Using (3), B. C., at $x = 0$ becomes

\[
a_{11}Y(0) + a_{12}Y(1) + b_{11}\sum_{k=0}^{n} b^k Y(k) + b_{12}\sum_{k=0}^{n} k b^{k-1} Y(k) = 0
\]

and B. C., at $x = b$ becomes

\[
a_{21}Y(0) + a_{22}Y(1) + b_{21}\sum_{k=0}^{n} b^k Y(k) + b_{22}\sum_{k=0}^{n} k b^{k-1} Y(k) = 0
\]

4. NUMERICAL ILLUSTRATIONS

The examples reported in this section were selected from a large collection of problems to which the differential transformation method could be applied. For purposes of comparison, contrast and performance, examples with known eigenvalues were chosen.

We use the absolute error which is defined as

\[
E_{DT} = |\lambda_{\text{exact solution}} - \lambda_{\text{differential transformation}}|
\]

Example 1: [11] We considered the following example

\[-u'' = \lambda u(x), \quad 0 < x < 1\]

and subject to the boundary conditions

\[u(0) + (\lambda - 4\pi^2)u'(0) = 0\]
\[u(1) - \lambda u'(1) = 0\]

The exact characteristic function is

\[
B_{\text{exact}} = \left(1 + 4\pi^2 - \lambda^2\right)^{\frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}}} - (2\lambda - 4\pi^2)\cos \sqrt{\lambda}
\]

where zero is not an eigenvalue.

Taking differential transformation of equation, we obtain

\[
Y(k + 2) = \frac{\lambda}{(k + 1)(k + 2)} Y(k)
\]

Using boundary condition (11), we have

\[
Y(0) + (\lambda - 4\pi^2)Y'(1) = 0
\]

or

\[
Y(1) = -\frac{Y(0)}{\lambda - 4\pi^2}
\]
\[ \sum_{k=0}^{n} Y(k) - \lambda \sum_{k=0}^{n} k Y(k) = 0 \quad (14) \]

Let

\[ Y(0) = c \]

we obtain

\[ Y(1) = -\frac{1}{\lambda - 4\pi^2} c \quad (15) \]

Following the same recursive procedure, we have

\[ Y(2) = -\frac{\lambda}{2!} c \]
\[ Y(3) = \frac{\lambda}{3(\lambda - 4\pi^2)} c \]
\[ Y(4) = \frac{\lambda^2}{4!} c \]
\[ Y(5) = \frac{\lambda^2}{5(\lambda - 4\pi^2)} c \]
\[ \vdots \]

\[ Y(n) = \begin{cases} (-1)^{\frac{n+1}{2}} \frac{\lambda^{\frac{n-1}{2}}}{n!(\lambda - 4\pi^2)} c, & n \text{ odd} ; \\ (-1)^{\frac{n}{2}} \frac{\lambda^{\frac{n}{2}}}{n!} c, & n \text{ even} . \end{cases} \]

Substituting (16) into (14), we obtain nonlinear equation in \( \lambda \). We can obtain \( \lambda_k \) by solving this equation. The higher values of \( n \) used, the more accurate the result. Table 1 lists the first three eigenvalues of example 1 at \( n = 40 \).

**Table 1** The first three eigenvalues in Example 1

<table>
<thead>
<tr>
<th>( \lambda_k )</th>
<th>( \lambda_{exact} )</th>
<th>( \lambda_{DT} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.73088657821308</td>
<td>9.73088657821309</td>
</tr>
<tr>
<td>2</td>
<td>88.7633162525897</td>
<td>88.7633162525991</td>
</tr>
<tr>
<td>3</td>
<td>157.884110438634</td>
<td>157.884110438597</td>
</tr>
</tbody>
</table>

Maximum absolute error are tabulated in Table 2 for Differential transformation together with the analogous results of B. Chanane [11].

**Table 2** Maximum absolute errors for Example 1

<table>
<thead>
<tr>
<th>( \lambda_k )</th>
<th>( PE_{DT} )</th>
<th>Results of [11]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0658E-014</td>
<td>1.1180E-06</td>
</tr>
<tr>
<td>2</td>
<td>9.3934E-012</td>
<td>7.8870E-05</td>
</tr>
<tr>
<td>3</td>
<td>3.7005E-011</td>
<td>1.1231E-04</td>
</tr>
</tbody>
</table>
We obtain the eigenfunction \( y_i(x) \) corresponding to eigenvalues \( \lambda_i \)

\[
y_i(x) = c \left[ 1 - \frac{1}{\lambda_i - 4\pi^2} x - \frac{\lambda_i}{2!} x^2 + \frac{\lambda_i^2}{3!(\lambda_i - 4\pi^2)} x^3 \\
+ \frac{\lambda_i^2}{4!} x^4 + \frac{\lambda_i^2}{5!(\lambda_i - 4\pi^2)} x^5 + \ldots \right]
\]

**Example 2:** [11] We considered the following example

\[-u'' + e^x u = \lambda u(x), \quad 0 < x < 1\]

and subject to the boundary conditions

\[
\begin{align*}
  u(0) &= 0 \\
  -\sqrt{\lambda} \sin \sqrt{\lambda} u(1) + \cos \sqrt{\lambda} u'(1) &= 0
\end{align*}
\]

The exact characteristic function is

\[
B_{\text{exact}} = i I_{2i\sqrt{\lambda}}(2) \left\{ \sqrt{\lambda} I_{-2i\sqrt{\lambda}} \left( 2\sqrt{e} \right) \sin \sqrt{\lambda} + \\
\left[ - \sqrt{\lambda} I_{2i\sqrt{\lambda}} \left( 2\sqrt{e} \right) \sin \sqrt{\lambda} \right] \\
- i I_{-2i\sqrt{\lambda}}(2) \left\{ \sqrt{\lambda} I_{2i\sqrt{\lambda}} \left( 2\sqrt{e} \right) \sin \sqrt{\lambda} \right.
\]

\[
\left. + \sqrt{e} \left[ I_{2i\sqrt{\lambda}} \left( 2\sqrt{e} \right) + i\frac{\sqrt{\lambda}}{\sqrt{e}} I_{-2i\sqrt{\lambda}} \left( 2\sqrt{e} \right) \right] \right\} \quad (17)
\]

Taking differential transformation of equation, we obtain

\[
Y(k + 2) = \frac{1}{(k + 1)(k + 2)} \left[ \sum_{L=0}^{k} \frac{1}{L!} Y(k - L) - \lambda Y(k) \right] \quad (18)
\]

and

\[
Y(0) = 0
\]

\[
- \sqrt{\lambda} \sin \sqrt{\lambda} \sum_{k=0}^{n} Y(k) + \cos \sqrt{\lambda} \sum_{k=0}^{n} k Y(k) = 0 \quad (19)
\]

Let

\[
Y(1) = c
\]

Following the same recursive procedure, we have
\[ Y(2) = 0, \]
\[ Y(3) = \frac{1}{6} (1 - \lambda)c, \]
\[ y(4) = \frac{1}{12} c, \]
\[ Y(5) = \frac{1}{30} \left[ 1 - \frac{1}{2} \lambda + \frac{1}{4} \lambda^2 \right] c, \]
\[ Y(6) = \frac{1}{24} \left[ 1 - \frac{1}{5} \lambda \right] c, \]
\[ Y(7) = \frac{1}{105} \left[ \frac{29}{48} - \frac{1}{3} \lambda + \frac{1}{16} \lambda^2 - \frac{1}{48} \lambda^3 \right] c, \]
\[ Y(8) = \frac{1}{56} \left[ \frac{1}{8} - \frac{1}{15} \lambda + \frac{1}{60} \lambda^2 \right] c, \]
\[ Y(9) = \frac{1}{18144} \left[ \frac{59}{4} - \frac{41}{5} \lambda + \frac{1}{5} \lambda^2 - \frac{1}{20} \lambda^3 + \frac{1}{21} \lambda^4 \right] c, \]
\[ Y(10) = \frac{1}{8640} \left[ \frac{103}{42} - \frac{43}{30} \lambda + \frac{1}{3} \lambda^2 - \frac{1}{21} \lambda^3 \right] c, \]
\[ \vdots \]

Substituting (20) into (19), we obtain nonlinear equation in \( \lambda \). We can obtain \( \lambda_k \) by solving this equation. The higher values of \( n \) used, the more accurate the result. \textbf{Table 3} lists the first three eigenvalues of example 2 at \( n = 40 \).

<table>
<thead>
<tr>
<th>( \lambda_k )</th>
<th>( \lambda_{exact} )</th>
<th>( \lambda_{DT} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.92906202857</td>
<td>0.92906202857890</td>
</tr>
<tr>
<td>2</td>
<td>6.7478811782</td>
<td>6.74788117821933</td>
</tr>
<tr>
<td>3</td>
<td>16.1245477258</td>
<td>16.1245477258556</td>
</tr>
</tbody>
</table>

Maximum absolute error are tabulated in \textbf{Table 4} for Differential transformation together with the analogous results of B. Chanane [11].

| \( \lambda_k \) | \(| E_{DT} |\) | Results of [11] |
|-----------------|------------------|------------------|
| 1               | 8.9000E-012      | 6.752E-09         |
| 2               | 1.9330E-011      | 3.717E-08         |
| 3               | 5.5600E-011      | 7.855E-08         |

The eigenfunction \( y_i(x) \) corresponding to eigenvalues \( \lambda_i \)

\[
y_i(x) = c \left[ x + \frac{1}{6} (1 - \lambda_i) x^3 + \frac{1}{12} x^4 + \frac{1}{30} \left[ 1 - \frac{\lambda_i}{2} + \frac{\lambda_i^2}{4} \right] x^5 \right. \\
+ \left. \frac{1}{24} \left( \frac{1}{3} - \frac{\lambda_i}{5} \right) x^6 + \frac{1}{105} \left[ \frac{29}{48} - \frac{\lambda_i}{3} + \frac{\lambda_i^2}{16} - \frac{\lambda_i^3}{48} \right] x^7 \right. \\
+ \left. \cdots \right] \\
+ \cdots
\]
\[
\frac{1}{56} \left[ \frac{1}{8} - \frac{\lambda_1}{15} + \frac{\lambda_1^2}{60} \right] x^8 + \frac{1}{1844} \left[ \frac{59}{4} - \frac{41}{5} \lambda_1 + \frac{1}{5} \lambda_1^2 - \frac{1}{40} \lambda_1^3 + \frac{1}{20} \lambda_1^4 \right] x^9 + \ldots
\]

**Example 3:** [12] In the case, \( \mu_0(x) \) is a complex-valued function satisfying \( \mu_0(x) \in L^1_{loc}(0,1) \) the equation (1) becomes

\[-u'' + e^{2ix} u(x) = \lambda u(x), \quad 0 \leq x \leq 1\]

and subject to the boundary conditions

\[u(0) + \sqrt{\lambda} u(1) = 0, \quad u'(0) = 0.\]

The exact characteristic function is

\[B_{exact} = \text{det} \begin{pmatrix}
J_{\sqrt{\lambda}}(1) + \sqrt{\lambda} J_{\sqrt{\lambda}}(e^i) & J_{-\sqrt{\lambda}}(1) + \sqrt{\lambda} J_{-\sqrt{\lambda}}(e^i) \\
\frac{J_{\sqrt{\lambda} - 1}(1) - J_{\sqrt{\lambda} + 1}(1)}{2} & \frac{J_{-\sqrt{\lambda} - 1}(1) - J_{-\sqrt{\lambda} + 1}(1)}{2}
\end{pmatrix}\]

Taking differential transformation of equation, we obtain

\[Y(k + 2) = \frac{1}{(k + 1)(k + 2)} \left[ \lambda \sum_{L=0}^{k} \frac{(2i)^L}{L!} Y(k - L) - \lambda Y(k) \right] \quad (21)\]

and

\[Y(1) = 0, \quad Y(0) + \sqrt{\lambda} \sum_{k=0}^{n} Y(k) = 0 \quad (22)\]

Following the same recursive procedure, we have

\[
\begin{align*}
Y(0) &= c \\
Y(1) &= 0 \\
Y(2) &= -\frac{1}{2}(\lambda - 1)c \\
Y(3) &= \frac{i}{3}c \\
Y(4) &= \frac{1}{24} \left[ \lambda^2 - 2\lambda - 3 \right] c \\
Y(5) &= -\frac{i}{15} \lambda c \\
Y(6) &= -\frac{1}{720} \left[ \lambda^3 - 3\lambda^2 - 25\lambda + 27 \right] c \\
Y(7) &= \frac{i}{280} \lambda^2 + \frac{13i}{1260} \lambda + \frac{79i}{2520} c \\
Y(8) &= \frac{1}{40320} \lambda^3 - \frac{1}{10080} \lambda^2 - \frac{41}{20160} \lambda^2 + \frac{1}{1440} \lambda + \frac{649}{40320} c \\
&\vdots
\end{align*}
\]

Substituting (23) into (22), we obtain nonlinear equation in \( \lambda \). The higher values of \( n \) used, the more accurate the
result. Table 5 lists the first three eigenvalues of example 4 at \( n = 40 \).

<table>
<thead>
<tr>
<th>( \lambda_k )</th>
<th>( \lambda_{\text{exact}} )</th>
<th>( \lambda_{\text{DT}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.9685430929 + 0.3906545895i</td>
<td>4.9685430929 + 0.3906545895i</td>
</tr>
<tr>
<td>2</td>
<td>20.6027103488 + 0.7502325235i</td>
<td>20.6027103488 + 0.7502325235i</td>
</tr>
<tr>
<td>3</td>
<td>64.140382447 + 0.68422837612i</td>
<td>64.140382447 + 0.68422837612i</td>
</tr>
</tbody>
</table>

The eigenfunction \( y_i(x) \) corresponding to eigenvalues \( \lambda_i \)

\[
y_i(x) = c \left[ 1 - \frac{1}{2} (\lambda_i - 1)x^2 + \frac{i}{3} x^3 + \frac{1}{24} \left( \lambda_i^2 - 2\lambda_i - 3 \right) \right] x^4
- \frac{i}{15} x^5 - \frac{1}{720} \lambda_i^3 - 3\lambda_i^2 - 25\lambda_i + 27 \right] x^6 + \left[ \frac{i}{280} \lambda_i^2 + \frac{13i}{1260} \lambda_i + \frac{79i}{2520} \right] x^7 + \left[ \frac{1}{40320} \lambda_i^4 - \frac{1}{10080} \lambda_i^3 - \frac{41}{20160} \lambda_i^2 + \frac{1}{1440} \lambda_i + \frac{649}{40320} \right] x^8 + \ldots
\]

**Example 4:** [11] We considered the following example

\[-u'' = \lambda u(x), \quad 0 \leq x \leq 1 \]

and subject to the boundary conditions

\[
u(0) - 2u'(0) = 0
\]

\[(1 - \sqrt{\lambda})u(1) + (1 - \lambda)u'(1) = 0\]

The exact characteristic function is

\[B_{\text{exact}} = \left\{ 2 \cos \sqrt{\lambda} + \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} \right\} + \left\{ 1 - \sqrt{\lambda} \left[ -2\sqrt{\lambda} \sin \sqrt{\lambda} + \cos \sqrt{\lambda} \right] \right\}
\]

Taking differential transformation of equation, we obtain

\[Y(k + 2) = -\frac{\lambda}{(k + 1)(k + 2)} Y(k) \quad (24)\]

Using boundary condition at \( x = 0 \), we have

\[Y(0) - 2Y(1) = 0\]

or

\[Y(0) = 2Y(1)\]

Using boundary condition at \( x = 1 \), we have

\[\left( 1 - \sqrt{\lambda} \right) \sum_{k=0}^{n} Y(k) + (1 - \lambda) \sum_{k=0}^{n} k Y(k) = 0 \quad (25)\]

Let

\[Y(1) = c\]

Following the same recursive procedure, we have
\[ Y(0) = 2c \]
\[ Y(2) = -\lambda c \]
\[ Y(3) = -\frac{1}{6} \lambda c \]
\[ Y(4) = \frac{1}{12} \lambda^2 c \]
\[ Y(5) = \frac{1}{120} \lambda^2 c \]
\[ Y(6) = -\frac{1}{360} \lambda^3 c \]
\[ Y(7) = -\frac{1}{5040} \lambda^3 c \]
\[ Y(8) = \frac{1}{20160} \lambda^4 c \]
\[ Y(9) = \frac{1}{362880} \lambda^4 c \]
\[ \vdots \]

Substituting (26) into (25), we obtain nonlinear equation in \( \lambda \). The higher values of \( n \) used, the more accurate the result. Table 6 lists the first three eigenvalues of example 4 at \( n = 40 \).

**Table 6 The first three eigenvalues in Example 4**

<table>
<thead>
<tr>
<th>( \lambda_k )</th>
<th>( \lambda_{exact} )</th>
<th>( \lambda_{DR} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.929679054283188</td>
<td>0.929679054283189</td>
</tr>
<tr>
<td>2</td>
<td>9.9387434140</td>
<td>9.938743414030333</td>
</tr>
<tr>
<td>3</td>
<td>11.2738742105212</td>
<td>11.2738742105212</td>
</tr>
</tbody>
</table>

Maximum absolute error are tabulated in Table 7 for Differential transformation together with the analogous results of B. Chanane [11].

**Table 7 Maximum absolute errors for Example 4**

| \( \lambda_k \) | \( |E_{DR}| \) | Results of [11] |
|-----------------|-------------|----------------|
| 1               | 9.9920E-16  | 0.155E-07      |
| 2               | 3.0334E-11  | 4.875E-06      |
| 3               | 4.4408E-14  | 0.361E-08      |

The eigenfunction \( y_\lambda(x) \) corresponding to eigenvalues \( \lambda \)

\[ y_\lambda(x) = c \left[ 2 + x - \lambda_1 x^2 - \frac{1}{6} \lambda_1 x^3 + \frac{1}{12} \lambda_1^2 x^4 \\
+ \frac{1}{120} \lambda_1^2 x^5 - \frac{1}{360} \lambda_1^3 x^6 - \frac{1}{5040} \lambda_1^3 x^7 + \frac{1}{20160} \lambda_1^4 x^8 + \cdots \right] \]

5. **CONCLUSION**

The differential transformation technique is applied to solve a strongly sturm-liouville problems with parameter dependent boundary condition. We have presented a few examples to illustrate a method and compared the
computed eigenvalues with the exact ones obtained as the zeros of the exact characteristic functions. Numerical results are compared to those obtained by the regularized sampling method [11] to illustrate the effectiveness of the proposed method. It is shown that the differential transformation method is very promising in this problem. The results of Example 3 clearly indicate that our method are accurate even when the coefficients \( p_i \) is a complex-valued function satisfying \( p_i \in L^1_{loc} (0,1) \).

REFERENCES