SLIDING MODE CONTROL FOR A CLASS OF QUASI-LINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATION SYSTEMS WITH TIME-VARYING DELAY

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ABSTRACT
The sliding mode control (SMC) problem for a class of quasi-linear parabolic partial differential equation (PDE) systems with time-varying delay is considered. Firstly, the stability problem for the reduced order sliding dynamical equations is investigated and a sufficient condition for the stability of sliding motion is given. Then the SMC law, which forces the system state from any initial state to reach the sliding manifold within finite time, is designed. At last a simulation example is presented to illustrate effectiveness of the proposed method.

Keywords: Parabolic partial differential equation, Sliding mode control, Lyapunov functional, Time-varying delay

1. INTRODUCTION
The control problem of the quasi-linear parabolic PDE process has been considered by many authors [1-6]. A large number of practical processes, such as fluidized-bed and packed-bed reactors, coating processes, the chemical vapor deposition of thin films, and biological processes, etc, are typically modeled by quasi-linear parabolic PDE systems. The standard approximate techniques are generally applied to dynamic analysis and control design of these systems. Because parabolic PDE systems involve spatial differential operators, whose eigenspectrum can be partitioned into a finite dimensional slow one and an infinite dimensional stable fast one, one approximate method utilizes the eigenfunction expansion techniques to obtain an approximate ODE representation of the original PDE system, and then to design the controller for it [7-11]. Although this approach has been very useful in designing control systems for parabolic PDEs, it may require keeping a large number of modes to derive an order differential equation (ODE) model that yields the desired degree of approximation, leading to high dimensionality of the resulting controllers [12]. Furthermore, this approach does not provide any information about the discrepancy between the solutions of the PDE model and the reduced-order ODE model in finite time, which is essential for characterizing the transient performance of the closed-loop system.

Another approximate technique is discretization of the space domain of the original PDE systems. After discretization, many existing theories and techniques are available for lumped parameter systems. However, the order of the resulting system must be sufficiently high to obtain an acceptable accuracy [13,14]. The control performance using such approximate techniques can be limited if the approximate model is inaccurate due to a too low order. Therefore, an approximate model with a high order is desirable to achieve better control performance. But it is not practical for control design. Meanwhile, the use of the approximate model leads to take into account the presence of mismatch between the approximate model used for controller design and the actual process model. Motivated by these considerations, significant recent work has focused on the development of other controller synthesis of quasi-linear parabolic PDE systems. In order to develop the control performance, it is necessary that the accurate PDE models is used to deal with dynamic analysis and control design of these systems [15-17].

In addition, we know that a control design for PDE process must face the robust problem associated to uncertain nonlinearities and delayed measurements. However, although some control designs have a strong theoretical back up, there are not well developed results under the presence of uncertain nonlinearities and delayed measurements up to now.

SMC scheme is well known with its robustness against modeled dynamics, disturbances, time-delays and nonlinearities [18-22]. Furthermore, it has many good performances such as fast response, good transient performance, easy realization, and so on. The design procedure of a SMC system is a two-stage process. The first phase is to choose a set of sliding manifolds such that the original system restricted to the intersection of this sliding manifold has a desired behavior. The second phase is to design a SMC law that forces the system’s trajectories onto the sliding surface and maintains them on it. Due to the influence of sliding manifold on system stability and transient performance, the design of sliding manifold has become one of the main issues in the SMC. Some approaches had been proposed for the design of sliding manifold for nonlinear systems with time-delay. Normally, it is designed as a linear sliding manifold.
In this paper, the controller design problem of a class of quasi-linear parabolic PDE systems has been considered by using the SMC method. We consider the nonlinearity and the delayed measurement in system model. Firstly, a linear sliding manifold is designed and the reduced order sliding dynamical equation is derived. And using matrix norm inequalities and Lyapunov functional methods, a sufficient condition to guarantee the sliding motion to be stable is given. Then a unit SMC law that forces the system state to reach the sliding manifold from initial system state within finite time is directly synthesized. The proposed method is promising and easy to implement. A numerical example is shown to verify the effectiveness of the proposed method.

The online of this paper is as follows. The system model is described in Section 2. The main results are derived in Section 3 and Section 4. An illustrated example is presented in Section 5. Conclusions are given in the last section.

Notation: \( \| \cdot \|_2 \) denotes the Euclidean norm of a vector or its induce matrix norm. \( \| \cdot \|_2 \) denotes the norm on \( L^2(\Omega) \), i.e., \( \| f \|_2 = \int_\Omega \| f \|^2 dx \), where \( \Omega = \{ x \mid \| x \| < \sigma < +\infty \} \) is \( R^l \) is bounded region with the smooth boundary \( \partial \Omega \), \( \sigma \) is a known constant. \( \nabla \) indicates gradient operator. For the vector function \( f(x,t) = (f^1(x,t), \ldots, f^n(x,t))^T \in R^n \ (x \in \Omega, t \geq 0) \), \( \| f(x,t) \|_{L^2} = \left( \int_\Omega \sum_{i=1}^n (f^i(x,t))^2 dx \right)^{\frac{1}{2}} \).

\[
\nabla f(x,t) = \left( \nabla f^1(x,t), \nabla f^2(x,t), \ldots, \nabla f^n(x,t) \right)^T \quad \text{and} \quad \| \nabla f(x,t) \|_{L^2} = \left( \int_\Omega \sum_{i=1}^n \left( \frac{\partial}{\partial x_j} f^i(x,t) \right)^2 dx \right)^{\frac{1}{2}}.
\]

For matrix \( A \in R^{m \times n} \), \( \lambda_{\text{max}}(A) \) indicates its biggest eigenvalue.

2. SYSTEM MODEL DESCRIPTION

Consider the quasi-linear parabolic PDE systems of the form

\[
\begin{align*}
\frac{\partial p_j(x,t)}{\partial t} & = D \Delta p_j(x,t) + \sum_{j=1}^l F_j \frac{\partial p_j(x,t)}{\partial x_j} + A_{1j} p_1(x,t) + A_{2j} p_2(x,t) + A_{11} p_1(x,t - \tau(t)) + A_{12} p_2(x,t - \tau(t)) + f_j(p_1(x,t), p_2(x,t)) + Bu(x,t), \\
\frac{\partial p_1(x,t)}{\partial t} & = D \Delta p_1(x,t) + \sum_{j=1}^l F_j \frac{\partial p_1(x,t)}{\partial x_j} + A_{21} p_1(x,t) + A_{22} p_2(x,t) + \bar{A}_{21} p_1(x,t - \tau(t)) + f_1(p_1(x,t), p_2(x,t)) + Bu(x,t), \\
\frac{\partial p_2(x,t)}{\partial t} & = D \Delta p_2(x,t) + \sum_{j=1}^l F_j \frac{\partial p_2(x,t)}{\partial x_j} + A_{21} p_1(x,t) + A_{22} p_2(x,t) + \bar{A}_{21} p_1(x,t - \tau(t)) + f_2(p_1(x,t), p_2(x,t)) + Bu(x,t),
\end{align*}
\]

where \( p_j(x,t) = (p_1(x,t), \ldots, p^n(x,t))^T \in R^m \) and \( p_2(x,t) = (p^{n-1}(x,t), \ldots, p^n(x,t))^T \in R^m \) are the state vector, \( u(x,t) \in R^m \) is control input vector. \( x \in \Omega, t \geq 0 \) and \( \Omega = \{ x : \| x \| < \sigma < +\infty \} \) is \( R^l \) is bounded region with the smooth boundary \( \partial \Omega \). \( D \), \( \sigma \) and \( F_j, j = 1, \ldots, l \) are known positive constant, \( A_{1j} \in R^{(n-m)\times(n-m)}, A_{2j} \in R^{(n-m)\times m}, A_{21} \in R^{m\times(n-m)}, A_{22} \in R^{m\times m}, \bar{A}_{21} \in R^{(n-m)\times(n-m)}, \bar{A}_{22} \in R^{(n-m)\times m}, \) \( \Delta = \sum_{j=1}^l \Delta_j \) is laplacian operator on \( \Omega \).

The time-varying delay \( \tau(t) \) is a known function and satisfies the following conditions

\[
0 \leq \tau(t) \leq \tau
\]

and

\[
1 - \dot{\tau}(t) \geq h > 0.
\]

where \( \tau \) and \( h \) are known real positive constants.

The vector functions
and
\[ f_1(p_1(x,t), p_2(x,t)) = (f^1(p_1(x,t), p_2(x,t)) \ldots f^{n-m}(p_1(x,t), p_2(x,t)))^T \in \mathbb{R}^{n-m} \]

and
\[ f_2(p_1(x,t), p_2(x,t)) = (f^{n-m+1}(p_1(x,t), p_2(x,t)) \ldots f^n(p_1(x,t), p_2(x,t)))^T \in \mathbb{R}^m \]

are smooth. We suppose that the vector function \( f_1(p_1(x,t), p_2(x,t)) \) satisfies the following condition
\[
\|f_1(p(x,t))\| \leq \sqrt{\mu} \|p(x,t)\| \tag{4a}
\]
and
\[
\|f_2(p(x,t))\| \leq g(p(x,t))\|p(x,t)\|, \tag{4b}
\]
where \( \mu \) is known positive constants and function \( g(y) \) \((y \in \mathbb{R}^n)\) is known and satisfies \( g(y) > 0 \) for \( \forall y \in \mathbb{R}^n \).

The initial and boundary conditions of Eq. (1) are, respectively, as follows
\[ p_i(x,t) = \phi_i(x,t), \quad i = 1, 2, \quad (x,t) \in \partial \Omega \times [-\tau, +\infty), \tag{5} \]
\[ \frac{\partial p_i(x,t)}{\partial n} = 0, \quad i = 1, 2, \quad (x,t) \in \partial \Omega \times [\tau, +\infty) \tag{6} \]
where \( \phi_1(x,t) \in \mathbb{R}^{n-m} \) and \( \phi_2(x,t) \in \mathbb{R}^m \) are the appropriate smooth function on \( \Omega \). \( \vec{n} \) indicates outward unit normal of the boundary \( \partial \Omega \). In this paper, we mainly consider the SMC problem for PDE systems (1) with initial and boundary value conditions (5) and (6).

3. STABILITY OF SLIDING MOTION

In this section, we consider the stability problem of sliding motion of PDE systems (1) with (5) and (6). Select the sliding function as
\[ S(x,t) = C_1 p_1(x,t) + C_2 p_2(x,t), \tag{7} \]
where \( C_1 \in \mathbb{R}^{m(n-m)} \) and \( C_2 \in \mathbb{R}^{m(m)} \) are the sliding coefficient matrix to be designed later and
\[ S(x,t) = \begin{bmatrix} s_1(x,t) \\ s_2(x,t) \\ \vdots \\ s_{n-1}(x,t) \end{bmatrix} \in \mathbb{R}^n. \]
Correspondingly, select sliding manifold as \( S(x,t) = 0 \). When we design the sliding coefficient matrices \( C_1 \) and \( C_2 \), matrix \( C_2 \) is required to be invertible.

If the motion is confined on the sliding manifold \( S(x,t) = 0 \), it follows that
\[ p_2(x,t) = -C_2^{-1} C_1 p_1(x,t). \tag{8} \]

Denote that \( f_1(p_1(x,t), -C_2^{-1} C_1 p_1(x,t)) = \overline{f}_1(p_1(x,t)) \). Substituting (8) into the first form of Eq.(1), the sliding motion equation of PDE system (1) can be given as follows
\[
\frac{\partial p_1(x,t)}{\partial t} = D \Delta p_1(x,t) + \sum_{j=1}^l F_j \frac{\partial p_i(x,t)}{\partial x_j} + (A_{11} - A_{12} C_2^{-1} C_1) p_1(x,t) \\
+ (\overline{A}_{11} - \overline{A}_{12} C_2^{-1} C_1) p_1(x,t - \tau(t)) + \overline{f}_1(p_1(x,t)) \tag{9}
\]
From (4a), we have that \( \|f_1(p_1(x,t), p_2(x,t))\|_{L^2}^2 \leq \mu \|p_1(x,t)\|_{L^2}^2 + \|p_2(x,t)\|_{L^2}^2 \). Denote that \( \overline{\mu} = \mu(1 + \|C_2^{-1} C_1\|_{L^2}^2) \). Further one obtains \( \|\overline{f}_1(p_1(x,t))\|_{L^2}^2 \leq \overline{\mu} \|p_1(x,t)\|_{L^2}^2 \). Suppose that the sliding motion equation (9) satisfies the following initial value and boundary value condition.
\[ p_1(x,t) = \phi(x,t), \quad (x,t) \in \partial \Omega \times [-\tau, +\infty) \tag{10} \]
\[ \frac{\partial p_i(x,t)}{\partial n} = 0, \quad (x,t) \in \partial \Omega \times [\tau, +\infty). \] (11)

**Theorem 1.** Consider the sliding motion equation (9) with (10) and (11). If the following conditions 1) and 2) are satisfied,

1) \( D - \frac{1}{2} I - \frac{1}{2} > 0, -2D + |F|^2 < 0, \) \( (12) \)

2) \( \lambda_{\text{max}}((A_1 - A_2 C_2^{-1} C_1)^T + A_1 - A_2 C_2^{-1} C_1 + (2 + \frac{a}{h} + h + \mu) I) < -\lambda < 0, \) \( (13) \)

where \( |F| = \max_{1 \leq j \leq \Omega} \|F_j\|, \quad \alpha = \|A_{11} - A_{12} C_2^{-1} C_1\|^2 \) and \( \lambda \) is a known positive constant, then the solution of the sliding motion equation (9) with (10) and (11) is uniformly convergent to zero, i.e., \( \lim_{t \to \infty} p_i(x,t) = 0, \quad x \in \Omega. \)

**Proof.** Introduce Lyapunov functional

\[ N(t) = \frac{1}{2} h \int_\Omega (p_i(x,t))^T p_i(x,t) \, dx + \int_\Omega \int_{t-t(t)}^{t} a(p_i(x,s))^T p_i(x,s) \, ds \, dx. \]

Obviously, it is positive definite. Denote that \( G = A_1 - A_2 C_2^{-1} C_1, \quad \bar{G} = A_{11} - A_{12} C_2^{-1} C_1. \) Calculating the time derivative of functional \( N(t) \) along the trajectory of Eq. (9) obtains

\[ \frac{dN(t)}{dt} = Dh \int_\Omega (p_i(x,t))^T \partial p_i(x,t) \, dx + \int_\Omega \int_{t-t(t)}^{t} \bar{a}(p_i(x,s))^T p_i(x,s) \, ds \, dx. \]

Using Divergence Degree Theorem (see [23]) and boundary condition (11) gets

\[ \int_\Omega p_i^T(x,t) \Delta p_i(x,t) \, dx = -\sum_{i=1}^{n-m} \sum_{j=1}^{l} \int_\Omega \left( \frac{\partial p_j(x,t)}{\partial x_j} \right)^2 \, dx. \] (16)

In addition, we have

\[ \int_\Omega \sum_{j=1}^{l} F_j p_j(x,t) \frac{\partial p_j(x,t)}{\partial x_j} \, dx \leq \frac{1}{2} \sum_{j=1}^{l} \int_\Omega \left( p_j(x,t) \right)^2 \, dx + \int_\Omega \left( F_j \frac{\partial p_j(x,t)}{\partial x_j} \right)^2 \, dx. \]

\[ = \frac{1}{2} \int_\Omega \left( p_i(x,t) \right)^2 \, dx + \frac{1}{2} \sum_{j=1}^{l} F_j^2 \int_\Omega \left( \frac{\partial p_j(x,t)}{\partial x_j} \right)^2 \, dx. \] (17)

where \( i = 1, \ldots, n - m. \) Substituting (15), (16), (17) into (14) and using (13) get
\[
\frac{dN(t)}{dt} \leq -Dh \sum_{i=1}^{n-m} \sum_{j=1}^{l} \int_{\Omega} \left( \frac{\partial p_j^i(x,t)}{\partial x_j} \right)^2 dx + \frac{1}{2} h \sum_{i=1}^{n-m} \int_{\Omega} (p^i(x,t))^2 dx + \frac{1}{2} h \sum_{j=1}^{l} F_j^2 \int_{\Omega} \left( \frac{\partial p_j^i(x,t)}{\partial x_j} \right)^2 dx + \frac{1}{2} h \int_{\Omega} \left( p_j^i(x,t) p(x,t) + (\overline{f}_j(p_j^i(x,t)))^T \overline{f}_j(p_j^i(x,t)) \right) dx \\
+ \frac{1}{2} h \int_{\Omega} p_j^T(x,t)(G^T + G + (1 + \frac{a}{h}) I) p_j(x,t) dx \\
\leq -(D - \frac{1}{2} |F|^2) h \|\nabla p_j(x,t)\|^2_{L^2} + \frac{1}{2} h \lambda \|p_j(x,t)\|^2_{L^2}.
\]

Integrating the above form from \( T \) to \( t \) gets

\[
N(t) + (D - \frac{1}{2} |F|^2) h \int_T^t \|\nabla p_j(x,s)\|^2_{L^2} ds + \frac{1}{2} h \lambda \int_T^t \|p_j(x,s)\|^2_{L^2} ds \leq N(T).
\]

Obviously, \( N(T) \) is bounded. From the definition of functional \( N(t) \), one obtains \( N(t) \geq 0, \quad -\tau \leq t < \infty \). So from (12) one gets that \( N(t) \) is bounded, \( \int_T^\infty \|p_j(x,s)\|^2_{L^2} ds < \infty \) and \( \int_T^\infty \|\nabla p_j(x,s)\|^2_{L^2} ds < \infty \). These imply that \( \|p_j(x,s)\|^2_{L^2} \) and \( \|\nabla p_j(x,s)\|^2_{L^2} \) are bounded, \( \|p_j(x,s)\|^2_{L^2} \in L_4(0,\infty) \) and \( \|\nabla p_j(x,s)\|^2_{L^2} \in L_4(0,\infty) \).

Next we will prove that \( \|\nabla p_j(x,t)\|^2_{L^2} \) is bounded and \( \|\Delta p_j(x,t)\|^2_{L^2} \in L_4(0,\infty) \). Consider the time derivative of function \( \|\nabla p_j(x,t)\|^2_{L^2} \). Using Divergence Degree Theorem (see [23]) and boundary condition (11) gets

\[
\frac{1}{2} \frac{d}{dt} \|\nabla p_j(x,t)\|^2_{L^2} = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \sum_{i=1}^{n-m} \sum_{j=1}^{l} \left( \frac{\partial p_j^i(x,t)}{\partial x_j} \right)^2 dx \\
= \sum_{i=1}^{n-m} \nabla p_j^i(x,t) \cdot \frac{\partial}{\partial t} p_j^i(x,t) - \sum_{i=1}^{n-m} \int_{\Omega} \sum_{j=1}^{l} \frac{\partial^2}{\partial x_j^2} p_j^i(x,t) \frac{\partial}{\partial t} p_j^i(x,t) dx \\
= -\int_{\Gamma} (\Delta p_j(x,t))^T \frac{\partial}{\partial t} p_j(x,t) dx \\
= -D \|\Delta p_j(x,t)\|^2_{L^2} - \int_{\Omega} (\Delta p_j(x,t))^T \sum_{j=1}^{l} F_j \frac{\partial p_j(x,t)}{\partial x_j} dx \\
- \int_{\Omega} (\Delta p_j(x,t))^T (G p_j(x,t) + G p_j(x,t - \tau(t))) dx - \int_{\Omega} (\Delta p_j(x,t))^T \overline{f}_j(p_j(x,t)) dx \\
\tag{17}
\]

Similarly, the following forms easily follow.

\[
-\int_{\Omega} (\Delta p_j(x,t))^T G p_j(x,t) dx = -\sum_{i=1}^{n-m} \int_{\Omega} \Delta p_j^i(x,t) \sum_{k=1}^{n-m} G_{ik} p^k(x,t) dx \\
= -\sum_{i=1}^{n-m} \nabla p_j^i(x,t) \sum_{k=1}^{n-m} G_{ik} p^k(x,t) \cdot n_{\Omega} + \sum_{i=1}^{n-m} \int_{\Omega} \sum_{j=1}^{l} \frac{\partial}{\partial x_j} p_j^i(x,t) \sum_{k=1}^{n-m} G_{ik} \frac{\partial}{\partial x_j} p^k(x,t) dx \\
= \sum_{i=1}^{n-m} \int_{\Omega} (\nabla p_j^i(x,t))^T \nabla (\sum_{k=1}^{n-m} G_{ik} p^k(x,t)) dx \leq \frac{1}{2} \|\nabla p_j(x,t)\|^2_{L^2} + \frac{1}{2} (n-m) \|G\| \|\nabla p_j(x,t)\|^2_{L^2} \\
- \int_{\Omega} (\Delta p_j(x,t))^T \overline{G} p_j(x,t - \tau(t)) dx \leq \frac{1}{2} \|\Delta p_j(x,t)\|^2_{L^2} + \frac{1}{2} (n-m) \|\overline{G}\| \|\nabla p_j(x,t - \tau(t))\|^2_{L^2} \\
\tag{19}
\]

and

\[
-\int_{\Omega} (\Delta p_j(x,t))^T \sum_{j=1}^{l} F_j \frac{\partial p_j(x,t)}{\partial x_j} dx \leq \frac{1}{2} \|\Delta p_j(x,t)\|^2_{L^2} + \frac{1}{2} \|F\|^2 \|\nabla p_j(x,t)\|^2_{L^2},
\tag{20}
\]
where $[G] \triangleq \max_{i,k} \{ G_{ik}^2 \}$, $[\bar{G}] \triangleq \max_{i,k} \{ \bar{G}_{ik}^2 \}$. Based on the Poincare inequality, we have

$$- \int_\Omega \Delta p_1(x,t) f_1(p_1(x,t)) dx \leq \frac{1}{2} \| \Delta p_1(x,t) \|_{L^2}^2 + \frac{1}{2} \| \bar{p}_1(x,t) \|_{L^2}^2 \leq \frac{1}{2} \| \Delta p_1(x,t) \|_{L^2}^2 + \frac{1}{2} \rho \bar{\mu} \| \nabla p_1(x,t) \|_{L^2}^2,$$

(21)

where $\rho(\Omega)$ is a positive constant. Substituting (18), (19), (20), (21) into (17) gets

$$\frac{1}{2} \frac{d}{dt} \| \nabla p_1(x,t) \|_{L^2}^2 \leq -(D - \frac{1}{2} - \frac{1}{2}) \| \Delta p_1(x,t) \|_{L^2}^2 + (1 + \frac{1}{2} (n-m)) [G] + \frac{1}{2} \rho \bar{\mu} + \frac{1}{2} |F| \| \nabla p_1(x,t) \|_{L^2}^2 + \frac{1}{2} \rho \bar{\mu} \| \nabla p_1(x,t) \|_{L^2}^2.$$

Integrating both sides of the above form from $\tau$ to $t$ gets

$$\frac{1}{2} \| \nabla p_1(x,t) \|_{L^2}^2 + \int_\tau^t (D - \frac{1}{2} - \frac{1}{2}) \| \Delta p_1(x,s) \|_{L^2}^2 ds \leq \frac{1}{2} \| \nabla p_1(x,\tau) \|_{L^2}^2 + (1 + \frac{1}{2} (n-m)) [G] + \frac{1}{2} |F| \| \nabla p_1(x,s) \|_{L^2}^2 ds + \frac{1}{2} \rho \bar{\mu} \| \nabla p_1(x,\tau) \|_{L^2}^2.$$

Due to $\| \nabla p_1(x,t) \|_{L^2}^2 \in L_4(0, \infty)$, the right side of the above form is bounded. From (12), one obtains

$$D - \frac{1}{2} - \frac{1}{2} > 0.$$ So we obtain that $\| \nabla p_1(x,t) \|_{L^2}^2$ is bounded and $\| \Delta p_1(x,t) \|_{L^2}^2 \in L_4(0, \infty)$.

Next we prove that $\frac{d}{dt} \| p_1(x,t) \|_{L^2}^2 \in L_4(0, \infty)$ and $\frac{d}{dt} \| \Delta p_1(x,t) \|_{L^2}^2 \in L_4(0, \infty)$. From (22), we have

$$\| \Delta p_1(x,t) \|_{L^2}^2 \leq \frac{1}{2} \frac{d}{dt} \| \nabla p_1(x,t) \|_{L^2}^2 \leq (D - \frac{1}{2} - \frac{1}{2}) \| \Delta p_1(x,t) \|_{L^2}^2 + (1 + \frac{1}{2} (n-m)) [G] + \frac{1}{2} |F| \| \nabla p_1(x,t) \|_{L^2}^2 + \frac{1}{2} \rho \bar{\mu} \| \nabla p_1(x,t) \|_{L^2}^2.$$

Due to $\| \nabla p_1(x,t) \|_{L^2}^2 \in L_4(0, \infty)$ and $\| \nabla p_1(x,t) \|_{L^2}^2 \in L_4(0, \infty)$, we have

$$\int_\tau^t \| \frac{d}{ds} \| p_1(x,s) \|_{L^2}^2 \| ds < \infty.$$ This implies that $\frac{d}{dt} \| p_1(x,t) \|_{L^2}^2 \in L_4(0, \infty)$.

Now consider the time derivative of function $\| p_1(x,t) \|_{L^2}^2$. We have

$$\frac{1}{2} \frac{d}{dt} \| p_1(x,t) \|_{L^2}^2 = \int_\Omega (p_1(x,t))^T \frac{\partial}{\partial t} p_1(x,t) dx$$

$$= \int_\Omega (p_1(x,t))^T (D \Delta p_1(x,t) + \sum_{j=1}^l F_j \frac{\partial p_1(x,t)}{\partial x_j} + G p_1(x,t) + \bar{G} p_1(x,t - \tau(t)) + \bar{f}_1(p_1(x,t))) dx$$

$$= \int_\Omega (p_1(x,t))^T D \Delta p_1(x,t) dx + \int_\Omega (p_1(x,t))^T \sum_{j=1}^l F_j \frac{\partial p_1(x,t)}{\partial x_j} dx$$

$$+ \int_\Omega (p_1(x,t))^T G p_1(x,t) dx + \int_\Omega (p_1(x,t))^T \bar{G} p_1(x,t - \tau(t)) dx + \int_\Omega (p_1(x,t))^T \bar{f}_1(p_1(x,t)) dx.$$

The following form easily follows.

$$\int_\Omega (p_1(x,t))^T \bar{f}_1(p_1(x,t)) dx \leq \frac{1}{2} \| p_1(x,t) \|_{L^2}^2 + \frac{1}{2} \bar{\mu} \| p_1(x,t) \|_{L^2}^2 \leq \frac{1}{2} \bar{\mu} \| p_1(x,t) \|_{L^2}^2.$$

(23)
Consider PDE systems (1) with (5) and (6). The unit SMC law is designed as follows:
\[
\begin{align*}
  u(x,t) &= -(C_2B)^{-1}\left((C_1A_1 + C_2A_2)p_1(x,t) + (C_1A_{12} + C_2A_{22})p_2(x,t)\right) \\
          &- (C_2B)^{-1}\sum_{j=1}^{l} F_j \frac{\partial S(x,t)}{\partial x_j} \\
          &- (C_2B)^{-1}\frac{S(x,t)}{\|S(x,t)\|} \left(\sqrt{\mu \|C_1\| + g(p(x,t))}\|C_2\| \|\rho(x,t)\|\right) \\
          &- \alpha(C_2B)^{-1}S(x,t) - \rho(C_2B)^{-1}\frac{S(x,t)}{\|S(x,t)\|^2}.
\end{align*}
\]

where \(\alpha, \varepsilon, \rho\) and \(\sigma\) are positive constants selected properly. We now give the main result of this section.

**Theorem 2** Consider PDE systems (1) with (5) and (6). If
\[
\alpha - D > 0,
\]
then the SMC law (26) drives the PDE systems (1) with (5) and (6) onto the sliding manifold \(S(x,t) = 0\) within finite time.

**Proof** Notation \(\frac{\partial S(x,t)}{\partial t}\) denotes the time partial derivative of \(S(x,t)\) along the trajectory of (1). Then from (1) and (7), it follows that
\[
S^T(x,t) \frac{\partial S(x,t)}{\partial t} = S^T(x,t)(C_1 \frac{\partial p_1(x,t)}{\partial t} + C_2 \frac{\partial p_2(x,t)}{\partial t})
\]

\[
= DS^T(x,t) \Delta S(x,t) + S^T(x,t) \sum_{j=1}^{l} F_j \left(C_1 \frac{\partial p_1(x,t)}{\partial x_j} + C_2 \frac{\partial p_2(x,t)}{\partial x_j}\right)
\]

\[
+ S^T(x,t)((C_1A_1 + C_2A_2)p_1(x,t) + (C_1A_{12} + C_2A_{22})p_2(x,t))
\]

\[
+ S^T(x,t)((C_1A_1 + C_2A_2)p_1(x,t - \tau(t)) + (C_1A_{12} + C_2A_{22})p_2(x,t - \tau(t)))
\]

\[
+ S^T(x,t)(C_1f_1(p(x,t)) + C_2f_2(p(x,t))) + S^T(x,t)C_2Bu(x,t)
\]

where \(\frac{\partial S(x,t)}{\partial x_j} = \begin{bmatrix} S^1(x,t) \\
S^2(x,t) \\
\vdots \\
S^n(x,t) \end{bmatrix}\), \(j = 1, \ldots, l\). Substituting (26) into the above form gets
\[
S^T(x,t) \frac{\partial S(x,t)}{\partial t} = DS^T(x,t) \Delta S(x,t) - \alpha S^T(x,t)S(x,t) - \rho \|S(x,t)\|^{2-\sigma}.
\]

Denote that \(\Phi(x,t) = S^T(x,t) \Delta S(x,t) - S^T(x,t)S(x,t), \quad t \geq 0, \quad x \in \Omega\). Then one obtains
\[
S^T(x,t) \frac{\partial S(x,t)}{\partial t} \leq D\Phi(x,t) - (\alpha - D) \|S(x,t)\|^2 - \rho \|S(x,t)\|^{2-\sigma}.
\]

Next we prove that the inequality \(\Phi(x,t) \leq 0, \quad t \geq 0, \quad \forall x \in \Omega\) is valid. Let \(\Omega_1 = \{x \mid \Phi(x,t) > 0, \quad t \geq 0, \quad x \in \Omega\}\) and \(\Omega_2 = \{x \mid \Phi(x,t) \leq 0, \quad t \geq 0, \quad x \in \Omega\}\). We can prove that set \(\Omega_1\) is empty. Suppose set \(\Omega_1\) to be non-empty. Then we have that
\[
0 \leq \int_{\Omega_1} \Phi(x,t) \, dx = S^T(x,t)\nabla S(x,t) |_{\nabla \Omega_1} - \int_{\Omega_1} \nabla S^T(x,t) \nabla S(x,t) \, dx - \int_{\Omega} S^T(x,t)S(x,t) \, dx,
\]

where \(\nabla S(x,t) = \begin{bmatrix} \nabla S_1(x,t) \\
\vdots \\
\nabla S_n(x,t) \end{bmatrix}\). Since \(\partial \Omega_1 = (\partial \Omega \cap \partial \Omega_1) \cup (\Omega \cap \partial \Omega_1)\), then when \(x \in \partial \Omega \cap \partial \Omega_1\), it follows
from (6) that \( S^T(x,t)V S(x,t)\) \( \mid s=0 \). This contradicts to (28). If \( x \in \Omega \cup \partial \Omega_1 \), we have \( x \in \partial \Omega_1 \). Since \( \Phi(x,t) \leq 0 \) for \( x \in \partial \Omega_1 \), substituting this into (27) gets \( S^T(x,t)\frac{\partial S(x,t)}{\partial t} \leq -(\alpha - D) \|S(x,t)\|^2 \), \( t \geq 0 \), \( x \in \partial \Omega_1 \). This implies that \( \frac{1}{2} \frac{\partial}{\partial t} \left( \|S(x,t)\|^2 \right) = S^T(x,t)\frac{\partial S(x,t)}{\partial t} \leq -(\alpha - D) \|S(x,t)\|^2 \), \( t \geq 0 \), \( x \in \partial \Omega_1 \). Integrating the above form on \([0, t]\) gives \( \|S(x,t)\|^2 = \|S(x,0)\|^2 e^{-(\alpha - D)t} \), \( t \geq 0 \), \( x \in \partial \Omega_1 \). So we can choose a big enough real constant \( T_0 > 0 \) such that \( \|S(x,t)\|^2 \) is small enough when \( t \geq T_0 \), \( x \in \partial \Omega_1 \). Further \( \|s_i(x,t)\| \) is small enough, \( i = 1, \ldots, m \), \( t \geq T_0 \), \( x \in \partial \Omega_1 \). Since \( \Delta s_i(x,t) = s_i(x,t), i = 1, \ldots, m \), \( t \geq T_0 \), \( x \in \partial \Omega_1 \), we have \( \|\Delta s_i(x,t)\|, i = 1, \ldots, m \), is small enough when \( t \geq T_0 \), \( x \in \partial \Omega_1 \). Then \( \|\nabla s_i(x,t)\|, i = 1, \ldots, m \) is bounded when \( t \geq T_0 \), \( x \in \partial \Omega_1 \). This is contradicts to (28). Therefore, the set \( \Omega_1 \) is empty, i.e., \( \Phi(x,t) \leq 0 \) for \( \forall x \in \Omega \). From this, (27) can be returned into \( S^T(x,t)\frac{\partial S(x,t)}{\partial t} \leq -\rho \|S(x,t)\|^{2-\sigma} \).

Next we proof that after a finite time moment \( T \) the trajectory \( p(x,t,\varphi) \) from initial state \( \varphi(x,t) \) can reach the sliding manifold \( S(x,T) = 0 \) and stay on it thereby. Introduce notation \( Z(x,t) = \|S(x,t)\|^2 \). Then one obtains

\[
Z(x,t) \frac{2-\sigma}{2} \frac{\partial Z(x,t)}{\partial t} \leq -2\rho.
\]

Assume that after a finite time moment \( T \), the trajectory \( p(x,t,\varphi) \) from initial state \( \varphi(x,t) \) reaches the sliding manifold. Then it follows \( S(x,T) = 0 \). Integrating (29) from \( 0 \) to \( T \), one obtains

\[
\int_{Z(x,0)}^{Z(x,T)} (Z(x,t)) \left( \frac{2-\sigma}{2} \right) d(Z(x,t)) \leq -\int_{0}^{T} 2\rho dt = -2\rho T.
\]

Since \( \int_{Z(x,0)} Z(x,t) \left( \frac{2-\sigma}{2} \right) d(Z(x,t)) = \frac{1}{1+\frac{\sigma}{2}} (Z(x,t))^{1+\frac{\sigma}{2}} \bigg|_{Z(x,0)}^{Z(x,T)} \bigg( Z(x,T) \bigg)^{\frac{\sigma}{2}} \), the form (30) can be transformed into the following form \( (Z(x,T))^{\frac{\sigma}{2}} \geq \sigma \rho T \). Therefore we have

\[
T \leq \frac{1}{\sigma \rho} (Z(x,0))^{\frac{\sigma}{2}} = \frac{1}{\sigma \rho} \|S(x,0)\|^2 \leq \frac{1}{\sigma \rho} \|C_1 p_1(x,0) + C_2 p_2(x,0)\|^2,
\]

Thus, the SMC law (26) can drives the systems (1) onto the sliding manifold \( S(x,t) = 0 \) in finite time moment \( T \). The proof completes.

5. SIMULATION RESULTS

Consider the parabolic PDE systems (1) with \( n = 2 \), \( m = 1 \), \( l = 1 \), \( \sigma = 10 \), \( D = 1.2 \), \( F = -1 \), \( A_{11} = (-190) \), \( A_{12} = (160) \), \( A_{21} = (147) \), \( A_{22} = (-114) \), \( \bar{A}_{11} = (33) \), \( \bar{A}_{12} = (-112) \), \( \bar{A}_{21} = (-75) \), \( \bar{A}_{22} = (40) \), \( B = (1) \), \( f_1(p_1(x,t), p_2(x,t)) = 0.001 \sin(p_1(x,t) + p_2(x,t)) \), \( f_2(p_1(x,t), p_2(x,t)) = e^{1.75 p_1(x,t)} p_2(x,t) \), \( \mu = 0.05 \), \( \tau = 0.004 \), \( h = 0.6 \). Select sliding coefficient matrix \( C_1 = -1 \), \( C_2 = \frac{5}{2} \). By simply computing, we can obtain that \( D - \frac{1}{2} - \frac{1}{2} = 0.2 > 0 \), \( -2D + |F|^2 = -1.4 < 0 \), \( \bar{\mu} = 0.058 \).
\[ \lambda_{\text{max}} \left( (A_{11} - A_{12} C_2^{-1} C_1)^T + A_{11} - A_{12} C_2^{-1} C_1 + (2 + \frac{a}{h} + h\lambda + \mu I) \right) = -69.65 = -\lambda < 0.\] Thus the conditions 1) and 2) of Theorem 1 are satisfied. The sliding motion equation is
\[ \frac{\partial p_1(x,t)}{\partial t} = 1.2 \Delta p_1(x,t) - \frac{\partial p_1(x,t)}{\partial x} - 126 p_1(x,t) + 10.2 p_1(x,t - 0.004) + 0.001 \sin(3 p_1(x,t)) . \]

We select \( \alpha = 1.22, \rho = 3, \sigma = 3/2 \) and \( g(p(x,t)) = e^{\frac{p_1(x,t)}{1 + p_1(x,t)}} \), respectively. Then the SMC law \( u(x,t) \) is
\[ u(x,t) = -223 p_1(x,t) + 178 p_2(x,t) - 53 p_1(x,t - 0.004) - 84.8 p_2(x,t - 0.004) + 0.4 \frac{\partial S(x,t)}{\partial x} \]
\[ \frac{S(x,t)}{\|S(x,t)\|} \left( 0.0894 + e^{\frac{p_1(x,t)}{1 + p_1(x,t)}} \right) \left\| p(x,t) \right\| - 0.56 S(x,t) - 1.2 \frac{S(x,t)}{\|S(x,t)\|}^{0.2} . \]

Due to \( \alpha - D = 0.02 > 0 \), the condition of Theorem 2 is satisfied. The initial and boundary value conditions are given as follows:
\[ \phi_1(x,t) = (t + 0.0027)x^{1/4} \sin \left( \frac{\pi}{10} (x - 10) \right), \quad (x,t) \in (-10,10) \times [-0.004,0], \]
\[ \phi_2(x,t) = (t + 0.0021)x^{1/6} \cos \left( \frac{\pi}{8} (x - 10) \right)^2, \quad (x,t) \in (-10,10) \times [-0.004,0], \]
and
\[ \frac{\partial p_1(x,t)}{\partial x} = 0, \quad \frac{\partial p_2(x,t)}{\partial x} = 0, \quad x = \pm 10, \quad t \in [-0.004, +\infty) . \]

Simulation results are shown in Fig. 1-5. Fig. 1 and Fig. 2 present the state variables of the open-loop system with initial value and boundary value conditions, respectively. We can see that the open-loop system is unstable. Fig. 3 shows trajectory of the SMC law \( u(x,t) \). Fig. 4 shows the trajectory of the variable \( S(x,t) \). We see that the state of system reaches the sliding manifold \( S(x,t) = 0 \) within finite time. Fig. 5 shows the trajectory of the sliding motion. We see that the sliding motion of the systems is stable.

![Figure1. Trajectory of state variable p1(x,t) of the open-loop system](image-url)
Figure 2. Trajectory of state variable $p_2(x,t)$ of the open-loop system

Figure 3. Trajectory of SMC law $u(x,t)$
Figure 4. Trajectory of variable $S(x,t)$

Figure 5. Trajectory of the sliding motion
6. CONCLUSION
In this paper, the SMC problem for a class of quasi-linear parabolic PDE systems with time-varying delay has been investigated. A sufficient condition for the stability of the sliding motion has been given. The proposed SMC law ensures that the trajectory from any initial state reach the sliding manifold in finite time. At last, a numerical example has been given to verify the obtained results.

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8. REFERENCES