DARBOUX TRANSFORMATION AND EXPLICIT SOLUTIONS FOR TWO NEW DIFFERENTIAL-INTEGRAL EQUATIONS

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ABSTRACT
A Darboux transformation for two new differential-integral equations in the Manakov hierarchy is given by using the gauge transformation between the corresponding matrix spectral problems. As an application of the Darboux transformation, soliton solutions of the two equations are obtained.

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1. INTRODUCTION
In the past few years, there are extensive development in the area of nonlinear mathematical physics, especially in
the work of integrable systems. Fairly satisfied understanding has been got for nonlinear integrable models. To investigate the soliton equations numerically, which associated with the integrable systems closely, the problem that how to solve partial differential equations and get the explicit solutions of them becomes an important research topic in the meaning of theory and practical application. Especially the explicit solutions provide a powerful tool to study the variety of properties of the soliton equations. Fortunately, quite a few methods of finding solutions of such soliton equations are well established, such as the inverse scattering transformation [1],[2], the Hirota's method [3], the Bäcklund and Darboux transformations [4]-[7], the algebra-geometric method [8], and so on. Among the various approaches, the Darboux transformation is known to be a powerful tool for finding explicit solutions of soliton equations [9]-[14].

The main aim of the present paper is to construct a Darboux transformation and explicit solutions for the following two differential-integral equations

\[ u_t = u_x + w \partial^{-1}ur, \]
\[ v_t = v_x + r \partial^{-1}vw, \]
\[ w_t = 2w_x - u \partial^{-1}vw, \]
\[ r_t = 2r_x - v \partial^{-1}ur, \]

and

\[ u_t = w_x + w \partial^{-1}(uv - wr) - u \partial^{-1}vw, \]
\[ v_t = v \partial^{-1}vw, \]
\[ w_t = w \partial^{-1}vw, \]
\[ r_t = v_x - v \partial^{-1}(uv - wr) - r \partial^{-1}vw, \]

which were first proposed in [15]. This paper is organized as follows. In section 2, with the help of a gauge transformation of the \(3 \times 3\) matrix spectral problem, we construct a Darboux transformation of the nonlinear evolution equations (1) and (2). Then by using the resulting Darboux transformation, some explicit solutions of the equations (1) and (2) are presented from its trivial solutions in section 3.

2. DARBOUX TRANSFORMATION
It is known that Darboux transformation is a useful tool to solve soliton equations by which explicit solutions of quite a few soliton equations are obtained. Here we shall derive a Darboux transformation of (1) and (2) with the help of the gauge transformation between the matrix spectral problems. Next we firstly turn to the construction of the Darboux transformation for equation (1), which has a Lax pair \((u, v, w, r)\)
\[
\Phi_t = U(s, \lambda) \Phi = \begin{pmatrix}
-\lambda & u & w \\
v & \lambda & 0 \\
r & 0 & \lambda
\end{pmatrix},
\]

(3)

\[
\Phi_t = V(s, \lambda) \Phi = \begin{pmatrix}
-2\lambda & u & 2w \\
v & 0 & \partial^{-1}vw \\
2r & -\partial^{-1}ur & 2\lambda
\end{pmatrix},
\]

(4)

Now we assume that \( \psi^{(l)} = (\psi_1^{(l)}, \psi_2^{(l)}, \psi_3^{(l)})^T, 1 \leq l \leq 3 \), are three basic solutions of (3) and (4), from which a fundamental matrix of solutions is defined by \( \Psi = (\psi^{(1)}, \psi^{(2)}, \psi^{(3)}) \). For the time being we introduce a gauge transformation for the spectral problems (3) and (4), \( \Psi \to \overline{\Psi} : \)

\[
\overline{\Psi} = T\Psi = \begin{pmatrix}
b_{11} + \lambda & b_{12} & b_{13} \\
b_{21} & b_{22} + \lambda & b_{23} \\
b_{31} & b_{32} & b_{33} + \lambda
\end{pmatrix},
\]

(5)

where \( b_i, i, j = 1, 2, 3 \) are determined later. A direct calculation gives rise to

\[
\det T = \lambda^3 + (b_{11} + b_{22} + b_{33})\lambda^2 + \cdots.
\]

Let \( \lambda_1, \lambda_2, \lambda_3 \) (pairwise distinct) be three arbitrary given parameters and the roots of third-order polynomial \( \det T \).

Then the column vectors of \( \overline{\Psi} \) are linearly dependent as \( \lambda = \lambda_j \) (\( j = 1, 2, 3 \)), which implies a linear algebraic system

\[
\begin{align*}
b_{11} + \lambda_j + \sigma_1^{(j)}b_{12} + \sigma_2^{(j)}b_{13} &= 0, \\
b_{21} + \sigma_1^{(j)}(b_{22} + \lambda_j) + \sigma_2^{(j)}b_{23} &= 0, \\
b_{31} + \sigma_1^{(j)}b_{32} + \sigma_2^{(j)}(b_{33} + \lambda_j) &= 0
\end{align*}
\]

(6)

with \( \gamma_k^{(j)} \) are dependent coefficients

\[
\sigma_1^{(j)} = \frac{\psi_2^{(1)}(\lambda_j) + \gamma_1^{(j)}\psi_2^{(2)}(\lambda_j) + \gamma_2^{(j)}\psi_2^{(3)}(\lambda_j)}{\psi_1^{(1)}(\lambda_j) + \gamma_1^{(j)}\psi_1^{(2)}(\lambda_j) + \gamma_2^{(j)}\psi_1^{(3)}(\lambda_j)}, \quad \sigma_2^{(j)} = \frac{\psi_3^{(1)}(\lambda_j) + \gamma_1^{(j)}\psi_3^{(2)}(\lambda_j) + \gamma_2^{(j)}\psi_3^{(3)}(\lambda_j)}{\psi_1^{(1)}(\lambda_j) + \gamma_1^{(j)}\psi_1^{(2)}(\lambda_j) + \gamma_2^{(j)}\psi_1^{(3)}(\lambda_j)},
\]

where \( \sigma_1^{(j)} \) satisfy four Riccati equations

\[
\sigma_1^{(j)} = v + 2\lambda_j \sigma_1^{(j)} - u(\sigma_1^{(j)})^2 - w\sigma_1^{(j)}\sigma_2^{(j)}, \quad \sigma_2^{(j)} = r + 2\lambda_j \sigma_2^{(j)} - u\sigma_1^{(j)}\sigma_2^{(j)} - w(\sigma_2^{(j)})^2
\]

and

\[
\sigma_1^{(j)} = v + 2\lambda_j \sigma_1^{(j)} + \partial^{-1}(vw)\sigma_1^{(j)} - u(\sigma_1^{(j)})^2 - 2w\sigma_1^{(j)}\sigma_2^{(j)}, \quad \sigma_2^{(j)} = 2r - \partial^{-1}(ur)\sigma_1^{(j)} + 4\lambda_j \sigma_2^{(j)} - u\sigma_1^{(j)}\sigma_2^{(j)} - 2w(\sigma_2^{(j)})^2.
\]

If parameters \( \lambda_j \) and \( \gamma_k^{(j)} \) are suitably chosen such that the determinant of the coefficients for (6) are nonzero, then \( b_{ij} \) are uniquely determined by the linear algebraic system (6).

We now introduce a linear transformation

\[
\overline{\Phi} = T\Phi,
\]

(7)

by which (3) and (4) are transformed into two spectral problems of the same type in the case \( \lambda \neq \lambda_j \) as follows

\[
\overline{\Phi}_s = \overline{U}(s, \lambda)\overline{\Phi}, \quad \overline{\Phi}_t = \overline{V}(s, \lambda)\overline{\Phi},
\]

(8)
where \( s = (\bar{u}, \bar{v}, \bar{w}, \bar{r}) \),
\[
\begin{align*}
\bar{U} &= (T_x + TU)^{-1}, \quad \bar{V} = (T_r + TV)^{-1}.
\end{align*}
\] (9)
It turns out that \( \lambda = \lambda_j \) are removable isolated singularities of \( \bar{U} \) and \( \bar{V} \) (see below). Therefore, we can define \( \bar{U} \) and \( \bar{V} \) for all \( \lambda \neq 0 \) by analytic continuation. Equation (9) can be written as
\[
(T_x + TU)^* = F = (f_{st}(\lambda))_{3 \times 3}
\] (10)
\[
(T_r + TV)^* = G = (g_{st}(\lambda))_{3 \times 3}
\] (11)
with \( T^{-1} = T^*/\det T \). A direct calculation gives that
\[
(T_x + TU) = \begin{pmatrix}
h_{11} & b_{12} + u(\lambda + b_{11}) + \lambda b_{12} & b_{13} + w(\lambda + b_{11}) + \lambda b_{13} \\
h_{21} & b_{22} + u b_{21} + \lambda(\lambda + b_{22}) & b_{23} + w b_{21} + \lambda b_{23} \\
h_{31} & b_{32} + u b_{31} + \lambda b_{32} & b_{33} + w b_{31} + \lambda(\lambda + b_{33})
\end{pmatrix},
\] (12)
where
\[
\begin{align*}
h_{11} &= b_{11} - \lambda(\lambda + b_{11}) + v b_{12} + r b_{13}, \\
h_{21} &= b_{21} - \lambda(\lambda + b_{21}) + v(\lambda + b_{22}) + r b_{23}, \\
h_{31} &= b_{31} - \lambda b_{31} + v b_{32} + r(\lambda + b_{33}).
\end{align*}
\]
It is not difficult to verify from (10), (12) and the Riccati equations that \( f_{23} = f_{32} = 0 \), \( f_{12} = f_{13} = f_{21} = f_{31} = \det T \), and \( f_{11}, f_{22}, f_{33} \) are fourth-order polynomials of \( \lambda \). After a cumbersome calculation, we have that \( f_{st}(\lambda_j) = 0 \), \( 1 \leq s, l \leq 3, \quad j=1,2,3 \) in views of (10), (12) and the Riccati equations.

We thus obtain by comparing the coefficients of the same power of \( \lambda \) that
\[
(T_x + TU)^T = (\det T)P(\lambda)
\] (13)
with
\[
P(\lambda) = \begin{pmatrix}
-\lambda & u + 2 b_{12} & w + 2 b_{13} \\
v - 2 b_{21} & \lambda & 0 \\
r - 2 b_{31} & 0 & \lambda
\end{pmatrix},
\]
which immediately implies the following fact.

**Proposition 1.** The matrices \( \bar{U} \) and \( \bar{V} \) determined by (9) have the same forms as \( U \) and \( V \), that is
\[
\bar{U} = \begin{pmatrix}
-\lambda & \bar{u} & \bar{w} \\
\bar{v} & \lambda & 0 \\
\bar{r} & 0 & \lambda
\end{pmatrix}, \quad \text{and} \quad \bar{V} = \begin{pmatrix}
-2\lambda & \bar{u} & 2\bar{w} \\
\bar{v} & 0 & \partial^{-1}\bar{v}\bar{w} \\
2\bar{r} & 0 & 2\lambda
\end{pmatrix},
\]
in which original potentials \( u, v, w, r \) are mapped into new potentials \( \bar{u}, \bar{v}, \bar{w}, \bar{r} \) according to a transformation
\[
\bar{u} = u + 2 b_{12}, \quad \bar{v} = v - 2 b_{21}, \quad \bar{w} = w + 2 b_{13}, \quad \bar{r} = r - 2 b_{31}.
\] (14)

**Proof.** Noticing (11), we obtain that \( T_r + TV = (m_{ij})_{3 \times 3} \), where
In view of \( m_{ij} \) and differentiating (6) with respect to \( t \) and using the Riccati equations, we get 
\[
g_{sl}(\lambda_j) = 0, \quad 1 \leq s, l \leq 3, \quad j = 1, 2, 3.
\]
By comparing coefficients of the same powers of \( \lambda \) in \( g_{sl} \), we arrive at
\[
(T_t + TV)T^* = (\det T)Q(\lambda)
\]
with
\[
Q(\lambda) = \begin{pmatrix} q_{11}^{(0)} + q_{12}^{(0)} & q_{12}^{(0)} & q_{13}^{(0)} \\ q_{21}^{(0)} & 0 & q_{23}^{(0)} \\ q_{31}^{(0)} & q_{32}^{(0)} & q_{33}^{(0)} \end{pmatrix},
\]
where \( q_{sl}^{(0)} \) are functions of \( (x, t) \) independent of \( \lambda \) and
\[
q_{11}^{(0)} = -q_{33}^{(0)} = -2, \quad q_{12}^{(0)} = q_{33}^{(0)} = 0, \quad q_{12}^{(0)} = u + 2b_{12}, \quad q_{13}^{(0)} = 2w + 4b_{13}, \\
q_{21}^{(0)} = v - 2b_{21}, \quad q_{23}^{(0)} = 2b_{23} + \partial^{-1}vw, \quad q_{31}^{(0)} = 2r - 4b_{31}, \quad q_{32}^{(0)} = -2b_{32} - \partial^{-1}ur.
\]
Noting the coefficients of \( \lambda_j \) in (13), and (15), we find that \( \mathbf{V} \) is of the same form as \( \mathbf{V} \). This completes the proof of the proposition.

From proposition 1, we shall postulate the following theorem.

**Theorem 1.** Every solution \((u, v, w, r)\) of equation (1) is mapped into a new solution \((\mathbf{V}, \mathbf{V}, \mathbf{W}, \mathbf{F})\) of (1) under the Darboux transformation (14), where functions \( b_{12}, b_{21}, b_{13}, b_{31} \) are uniquely determined by (4).

In the similar way, we can state the next result.

**Theorem 2.** Under the Darboux transformation (14), every solution \((u, v, w, r)\) of equation (2) is mapped into a new one \((\mathbf{V}, \mathbf{V}, \mathbf{W}, \mathbf{F})\) of it, where the functions \( b_{12}, b_{21}, b_{13}, b_{31} \) are uniquely determined by (6).

### 3. Explicit Solutions

In this section, we shall apply the Darboux transformation to construct the explicit solutions of nonlinear evolution equations (1) and (2). Equation (6) can be rewritten as
\[
\begin{align*}
&\begin{align*}
&b_1 + \sigma_1^{(1)}b_2 + \sigma_2^{(1)}b_3 = -\lambda_1, \\
&b_1 + \sigma_1^{(2)}b_2 + \sigma_2^{(2)}b_3 = -\lambda_2, \\
&b_1 + \sigma_1^{(3)}b_2 + \sigma_2^{(3)}b_3 = -\lambda_3,
\end{align*} \\
&\begin{align*}
&b_1 + \sigma_1^{(1)}b_2 + \sigma_2^{(1)}b_3 = -\lambda_1, \\
&b_1 + \sigma_1^{(2)}b_2 + \sigma_2^{(2)}b_3 = -\lambda_2, \\
&b_1 + \sigma_1^{(3)}b_2 + \sigma_2^{(3)}b_3 = -\lambda_3,
\end{align*}
&b_1 + \sigma_1^{(1)}b_2 + \sigma_2^{(1)}b_3 = -\lambda_1, \\
&b_1 + \sigma_1^{(2)}b_2 + \sigma_2^{(2)}b_3 = -\lambda_2, \\
&b_1 + \sigma_1^{(3)}b_2 + \sigma_2^{(3)}b_3 = -\lambda_3,
\end{align*}
\]
which implies by the Cramer law that (we just display the four associated with the Darboux transformation)
\[
\begin{align*}
&b_{12} = \frac{\Delta_{12}}{\Delta}, \quad b_{13} = \frac{\Delta_{13}}{\Delta}, \quad b_{21} = \frac{\Delta_{21}}{\Delta}, \quad b_{31} = \frac{\Delta_{31}}{\Delta},
\end{align*}
\]
where
\[
\Delta = \begin{vmatrix} 1 & \sigma_1^{(1)} & \sigma_2^{(1)} \\ 1 & \sigma_1^{(2)} & \sigma_2^{(2)} \\ 1 & \sigma_1^{(3)} & \sigma_2^{(3)} \end{vmatrix}, \quad \Delta_{12} = \begin{vmatrix} -\lambda_1 & \sigma_2^{(1)} \\ -\lambda_2 & \sigma_2^{(2)} \\ -\lambda_3 & \sigma_2^{(3)} \end{vmatrix}, \quad \Delta_{13} = \begin{vmatrix} 1 & \sigma_1^{(1)} \\ 1 & \sigma_1^{(2)} \\ 1 & \sigma_1^{(3)} \end{vmatrix}, \quad \Delta_{21} = \begin{vmatrix} -\lambda_1 & \sigma_2^{(1)} \\ -\lambda_2 & \sigma_2^{(2)} \\ -\lambda_3 & \sigma_2^{(3)} \end{vmatrix}, \quad \Delta_{31} = \begin{vmatrix} 1 & \sigma_1^{(1)} \\ 1 & \sigma_1^{(2)} \\ 1 & \sigma_1^{(3)} \end{vmatrix}.
\]
\[ \Delta_{21} = \begin{vmatrix} -\lambda_1 \sigma_1^{(1)} & \sigma_1^{(1)} & \sigma_1^{(3)} \\ -\lambda_2 \sigma_1^{(2)} & \sigma_2^{(1)} & \sigma_2^{(3)} \\ -\lambda_3 \sigma_1^{(3)} & \sigma_3^{(1)} & \sigma_3^{(3)} \end{vmatrix}, \quad \Delta_{11} = \begin{vmatrix} -\lambda_1 \sigma_1^{(1)} & \sigma_1^{(1)} & \sigma_1^{(3)} \\ -\lambda_2 \sigma_2^{(2)} & \sigma_2^{(2)} & \sigma_2^{(3)} \\ -\lambda_3 \sigma_3^{(3)} & \sigma_3^{(3)} & \sigma_3^{(3)} \end{vmatrix}. \]

Here we assume that the parameters \( \lambda_1, \lambda_2, \lambda_3, \gamma_1^{(1)}, \gamma_1^{(2)}, \gamma_1^{(3)}, \gamma_2^{(1)}, \gamma_2^{(2)}, \gamma_2^{(3)} \) are suitably chosen, so that the determinant of coefficients \( \Delta \neq 0 \). Substituting (17) into (14) we arrive at the explicit form of the Darboux transformation

\[ \bar{u} = u + 2 \frac{\Delta_{12}}{\Delta}, \quad \bar{v} = v - 2 \frac{\Delta_{21}}{\Delta}, \quad \bar{w} = w + 2 \frac{\Delta_{13}}{\Delta}, \quad \bar{r} = r - 2 \frac{\Delta_{31}}{\Delta}. \] (18)

1) Choose a trivial solution \( u = 0, v = 0, w = 0, r = 0 \) of nonlinear evolution equation (1). Then (3) and (4) are reduced to

\[ \begin{align*}
\phi_{1,x} &= -\lambda \phi_1, & \phi_{2,x} &= \lambda \phi_2, & \phi_{3,x} &= \lambda \phi_3, \\
\phi_{1,t} &= -2\lambda \phi_1, & \phi_{2,t} &= 0, & \phi_{3,t} &= 2\lambda \phi_3.
\end{align*} \] (19)

Equation (19) has a fundamental matrix of solutions

\[ \Psi = \begin{pmatrix} e^{-\lambda x - 2\lambda t} & 0 & 0 \\
0 & e^{\lambda x} & 0 \\
0 & 0 & e^{\lambda x + 2\lambda t} \end{pmatrix}, \]

which together with (6) leads to

\[ \sigma_1^{(j)} = \gamma_1^{(j)} e^{2\lambda x + 2\lambda j t}, \quad \sigma_2^{(j)} = \gamma_2^{(j)} e^{2\lambda x - 4\lambda j t}, \quad j = 1,2,3. \]

Therefore, we obtain by the Darboux transformation (18) that an explicit solution of (1)

\[ \begin{align*}
\bar{u} &= \frac{2}{\Upsilon_1} \left[ (\lambda_2 - \lambda_3) \gamma_2^{(1)} e^{B_1} + (\lambda_3 - \lambda_1) \gamma_2^{(2)} e^{B_2} + (\lambda_1 - \lambda_2) \gamma_2^{(3)} e^{B_3} \right], \\
\bar{v} &= \frac{\Upsilon_1}{2} \left[ (\lambda_2 - \lambda_3) \gamma_2^{(1)} e^{A_2 + A_3 + B_1} + (\lambda_3 - \lambda_1) \gamma_2^{(2)} e^{A_1 + A_3 + B_1} + (\lambda_1 - \lambda_2) \gamma_2^{(3)} e^{A_1 + A_2 + B_1} \right], \\
\bar{w} &= \frac{2}{\Upsilon_1} \left[ (\lambda_3 - \lambda_2) \gamma_2^{(1)} e^{A_1} + (\lambda_1 - \lambda_3) \gamma_2^{(2)} e^{A_2} + (\lambda_2 - \lambda_1) \gamma_2^{(3)} e^{A_3} \right], \\
\bar{r} &= \frac{\Upsilon_1}{2} \left[ (\lambda_3 - \lambda_2) \gamma_2^{(1)} e^{A_1 + B_2 + B_3} + (\lambda_1 - \lambda_3) \gamma_2^{(2)} e^{A_1 + B_2 + B_3} + (\lambda_2 - \lambda_1) \gamma_2^{(3)} e^{A_2 + B_2 + B_3} \right],
\end{align*} \] (20)

where \( (A_j = 2\lambda_j x + 2\lambda_j t, \quad B_j = 2\lambda_j x + 4\lambda_j t, \quad j = 1,2,3) \)

\[ \Upsilon_1 = \gamma_2^{(1)} \gamma_2^{(3)} e^{A_2 + B_3} + \gamma_1^{(1)} \gamma_2^{(2)} e^{A_1 + B_2} + \gamma_2^{(1)} \gamma_1^{(3)} e^{A_3 + B_1} - \gamma_1^{(2)} \gamma_2^{(1)} e^{A_2 + B_1} - \gamma_3^{(3)} \gamma_2^{(2)} e^{A_3 + B_2} - \gamma_1^{(1)} \gamma_2^{(3)} e^{A_1 + B_3}. \]

The plans of the solution (20) are given in Figures 1-4.
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Figure 1-4: The solutions \( \tilde{H}, \tilde{V}, \tilde{W}, \tilde{F} \) with

\[
k_1 = -0.15, k_2 = -0.15, k_3 = 0.2, \gamma^{(1)} = -3, \gamma^{(2)} = -4, \gamma^{(3)} = -0.1, \gamma^{(2)} = -0.001, \gamma^{(3)} = 0.01, \gamma^{(2)} = -0.15.
\]

2) It is obvious that \( u = 0, v = 0, w = 0, r = 0 \) are solutions of equation (2). In the case, (3) and (4) are reduced to

\[
\begin{align*}
\phi_{1,x} &= -\lambda \phi_1, & \phi_{2,x} &= \lambda \phi_2, & \phi_{3,x} &= \lambda \phi_3, \\
\phi_1 &= \phi_2 = 0, & \phi_3 &= 2\lambda \phi_2.
\end{align*}
\]

Equation (21) has a fundamental matrix of solutions

\[
\Psi = \begin{pmatrix}
e^{-\lambda x} & 0 & 0 \\
0 & e^{\lambda x} & 0 \\
0 & 2\lambda te^{\lambda x} & e^{\lambda x}
\end{pmatrix},
\]

which together with (6) leads to

\[
\sigma^{(j)} = \gamma^{(j)} e^{2\lambda j x}, \quad \sigma^{(j)} = 2\lambda \gamma^{(j)} e^{2\lambda j x} + \gamma^{(j)} e^{2\lambda j x}, \quad j = 1, 2, 3.
\]

Therefore, we obtain by the Darboux transformation (18) that an explicit solution of (2)

\[
\begin{align*}
\tilde{u} &= \frac{2}{\gamma_2} \left[ (\lambda_2 - \lambda_3)(\lambda_3 t \gamma_1^{(1)} + \gamma_2^{(1)}) e^{C_1} + (\lambda_3 - \lambda_1)(2\lambda_2 t \gamma_1^{(2)} + \gamma_2^{(2)}) e^{C_2} \\
&\quad + (\lambda_1 - \lambda_2)(2\lambda_3 t \gamma_1^{(3)} + \gamma_2^{(3)}) e^{C_3} \right], \\
\tilde{v} &= \frac{2}{\gamma_2} \left[ (\lambda_3 - \lambda_2) \gamma_1^{(1)} e^{C_1} + (\lambda_1 - \lambda_3) \gamma_1^{(2)} e^{C_2} + (\lambda_2 - \lambda_1) \gamma_1^{(3)} e^{C_3} \right], \\
\tilde{w} &= \frac{2}{\gamma_2} \left[ (\lambda_3 - \lambda_1) \gamma_1^{(1)} + (\lambda_2 - \lambda_1) \gamma_1^{(2)} + (\lambda_1 - \lambda_2) \gamma_1^{(3)} \right], \\
\tilde{r} &= \frac{2}{\gamma_2} \left[ (\lambda_3 - \lambda_1) \gamma_1^{(1)} (2\lambda_1 t \gamma_1^{(1)} + \gamma_2^{(1)}) (2\lambda_3 t \gamma_1^{(3)} + \gamma_2^{(3)}) \\
&\quad + (\lambda_3 - \lambda_2) \gamma_1^{(1)} (2\lambda_2 t \gamma_1^{(2)} + \gamma_2^{(2)}) (2\lambda_3 t \gamma_1^{(3)} + \gamma_2^{(3)}) \\
&\quad + (\lambda_2 - \lambda_1) \gamma_1^{(2)} (2\lambda_1 t \gamma_1^{(1)} + \gamma_2^{(1)}) (2\lambda_2 t \gamma_1^{(2)} + \gamma_2^{(2)}) \right],
\end{align*}
\]

where \( C_j = 2\lambda_j x, \quad j = 1, 2, 3 \)

\[
\begin{align*}
\gamma_2 &= \left[ \gamma_1^{(2)} (\gamma_1^{(3)} - 2\lambda_2 t \gamma_1^{(3)}) - \gamma_1^{(3)} (\gamma_2^{(2)} - 2\lambda_3 t \gamma_1^{(2)}) \right] e^{C_2} + C_3 + \left[ \gamma_1^{(1)} (\gamma_2^{(2)} - 2\lambda_1 t \gamma_1^{(2)}) \\
&\quad - \gamma_1^{(2)} (\gamma_1^{(3)} - 2\lambda_2 t \gamma_1^{(3)}) \right] e^{C_1} + C_2 + \left[ \gamma_1^{(3)} (\gamma_1^{(2)} - 2\lambda_3 t \gamma_1^{(2)}) - \gamma_1^{(1)} (\gamma_2^{(3)} - 2\lambda_1 t \gamma_1^{(3)}) \right] e^{C_1} + C_3.
\end{align*}
\]

The solution curves of the solution (22) are given in Figures 5-8.
Figure 5-8: The solution $\bar{u}, \bar{v}, \bar{w}, \bar{r}$ with

$$k_1 = -0.15, k_2 = -0.15, k_3 = 0.2, \gamma_1^{(1)} = -3, \gamma_2^{(1)} = -4, \gamma_1^{(2)} = -0.1, \gamma_2^{(2)} = -0.001, \gamma_1^{(3)} = 0.01, \gamma_2^{(3)} = -0.15.$$ 

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