RECURSIVE METHOD FOR CONSTRUCTION OF RESOLVABLE NESTED DESIGNS AND UNIFORM DESIGNS ASSOCIATED

A.Boudraa\textsuperscript{1}, Z.Gheribi-Aoulmi\textsuperscript{2} & M. Laib\textsuperscript{3} \\
\textsuperscript{1,2,3}Laboratory M.A.M, University Constantine 1, Algeria. \\
E-mail: \textsuperscript{1}A.Boudraa: abla-boudraa@hotmail.fr \\
\textsuperscript{2}Z.Gheribi-Aoulmi: gheribiz@yahoo.fr \\
\textsuperscript{3}M.Laib: laib.med@gmail.com

ABSTRACT
This paper presents a recursive method for the construction of resolvable balanced incomplete nested block design. The union of the design on stage \((n)\) provides a resolvable balanced incomplete block design: \(R_n^*(v, b, k, \lambda)\), from which also another resolvable design: \(Q_n^*(v, b, r, k, \lambda)\) more economic in blocks number and treatments repetitions can be obtained for use. Moreover, a series of symmetrical uniform design with \(r_n^*\) factors can be deducted and on which the number of levels function on the stage of recurrence \((n)\) chosen, \(p\) is order of a Galois fields \(GF(p)\). These designs denote by: \(U(v, p)\). The configuration of designs \((BIB, Resolvable\ and\ Uniform)\), building by our method, can be obtained using the package \texttt{R PGM2} that we have realized and which is available either upon request by e-mail on: laib.med@gmail.com or on http://cran.r-project.org.

Classification AMS 2000: primary 05B05; secondary 62K10.

Keywords: Balanced incomplete binary blocks, resolvable designs, finite projective geometry, finite linear sub-variety, uniform design

1. INTRODUCTION
A balanced incomplete binary blocks design \((BIB)\) with parameters \(v, b, r, k, \lambda\) is defined as an arrangement of \(v\) treatments into \(b\) blocks of \(k\) \((k < v)\) treatments each such that:
1. Each treatments occurs at most once in a block,
2. Each treatment occurs exactly \(r\) different blocks,
3. Every pair of treatments occurs together in exactly \(\lambda\) blocks.

There are several methods for the \texttt{BIB} designs construction (cf. Raghavaro [15]). The well-known construction method that we use to build the designs that we propose, consists in its identification with a system of linear sub-varieties of an \(m\) -dimensional projective geometry \(PG(m, p)\), denoted by \(V_m\), defined on a Galois fields \(GF(p)\) of \(p\) elements. This identification consists to present a treatment as a point of coordinates \((x_1, x_2, ..., x_m)\) of this geometry and a block as an \(h\) -dimensional linear sub-variety \((h < m)\).

It is known that \(v r = bk\), \(\lambda(v - 1) = r(k - 1)\) and \(v \leq b\) hold. In particular, when \(b = v\), the \texttt{BIB} design is said to be symmetric. A \texttt{BIB} design is called resolvable (cf. Bose [4]) if the blocks of the \texttt{BIB} design can be divided into \(r\) groups such that each group contains each of the \(v\) treatments exactly once. The recent uses of resolvable block design as bases designs for the construction of uniform design in numerical experiments [6, 7, ..., etc.] benefit them.

A symmetric uniform design is a type of "space filling" design for computer experiments (cf. Fang [8]), denoted by \(U(v, p)\). Exactly, It is an \(v \times r\) matrix \(U_{v,r} = (u_{ij})\) with entries 1,..., \(p\) in the \(j\)\textsuperscript{th} column \((j = 1, ..., r)\), such that each entry in each column appears the same number of times.

The method developed in this paper is characterized by the fact that it recurring and that it provides a series of
resolvable nested block design on each stage of the recurrence, consequently, it allow the obtaining of symmetrical uniform design where levels of factors are the \( n^{th} \) power of \( p \) (\( n \) being the stage of recurrence).

2. CONSTRUCTION

Let a \( m \)-dimensional projective geometry defined on a Galois field of order \( p \), \( V_m \).

**Step 1**

- Build the set of all \( m_i \)-dimensional distinct linear sub-varieties (\( m_i = m - 1 \)):
  \[
  \{V(i_1) : 1 \leq i_1 \leq b_1\} \text{ in } V_m, \text{ by resolving the equations system: } \sum_{u=0}^{u=m} a_{1u} x_u = 0 \text{ (The } a_{1u} \text{ are reduced } \mod(p) \text{ and } x_u \in GF(p)).
  \]
- Identify the system \( \{V(i_1) : 1 \leq i_1 \leq b_1\} \) with a system of BIB design, said of \( 1^{st} \) generation.

**Step 2**

Fix \( n \) (\( 1 < n < m \)) and successively build the set of all \( m_j \)-dimensional nested sub-varieties (\( m_j = m - j \), \( j = 2, ..., n \)):
\[
\{V(i_1, i_2, ..., i_j) : 1 \leq i_j \leq b_j\} \text{ in } V(i_1, i_2, ..., i_{j-1}) \text{ while operating as at Step 1, that is to say, by solving the of the equations system:}
\]
\[
\left\{
\begin{align*}
\sum_{u=0}^{u=m} a_{1u} x_u &= 0 \\
\sum_{u=0}^{u=m} a_{ju} x_u &= 0
\end{align*}
\right.
\]
where the \( a_{1u} \) and \( a_{ju} \in GF(2) \) and \( l = j - 1 \).

Indeed, all \( m_j \)-dimensional sub-varieties in \( m_{j-1} \)-dimensional projective geometry is a \( m_j \)-dimensional projective geometry. The BIB associates \( \{V(i_1, i_2, ..., i_j) : 1 \leq i_j \leq b_j\} \) are said \( j^{th} \) generation.

**Step 3**

- Delete any sub-varity \( V(i_1) \) of first generation, defined by an equation of the type: \( \sum_{u=0}^{u=m} a_{1u} x_u = 0 \) and all sub-varieties generated by this one at the second step and downward sub-varieties.
- Delete all \( m_j \)-dimensional sub-varieties (\( m_j \leq m_2 \), \( j = 2, ..., n \)) included in the system of the equations defining them, the \( V(i_j) \) equation.
- Remove the points belonging to \( V(i_j) \) and existing in others sub-varieties, regardless of their dimension.

The union of all sub-varieties at each stage \( j = 2, ..., n \) provides a resolvable design to repeated blocks \( \alpha \) times.

\[
\alpha = \prod_{i=2}^{i=j} (1 + p + p^2 + ... + p^{i-1})
\]

**Example 1** Let \( PG(3,2) \), noted \( V_3 \). By step 1, we obtain the first generation BIB: \( \{V(i_1) : 1 \leq i_1 \leq 15\} \) with the parameters: \( v_1 = b_1 = 15, r_1 = k_1 = 7 \) and \( \lambda_1 = 3 \) (see Figure 1). By step 2, we obtain a series of \( 2^{nd} \) generation blocks designs BIB:
\[
\{V(i_1, i_2) : 1 \leq i_2 \leq 7\} \text{ in } V(i_1) \text{ and } 1 \leq i_1 \leq 15. \]
Each BIB has the parameters:
\( v_2 = b_2 = 7, r_2 = k_2 = 3 \) and \( \lambda_2 = 1 \).

The steps of the stage 3 is illustrated by Figure 2, considering sub-varity \( V(i_1) \) defined by the equation

\[ x_0 = 0. \]

The union of all remaining blocks is a resolvable design and each block is repeated 3 times.
Figure 1: Illustration of the step 1 and 2.

Figure 2: Illustration of the step 3.
For further argument, the following results (cf. Dugué p. 280-284 [5]) are useful:

**Result 1** Let $V_m$ an $m$-dimensional projective geometry defined on a Galois field of order $p$. The number of $h$-dimensional distinct sub-varieties $V_m$ passing by a fixed $l$-dimensional sub-variety ($l < h < m$), is equal to:

$$N(m, h, l) = \frac{(p^{l+1} + p^{l+2} + \ldots + p^m)(p^{l+2} + \ldots + p^m)\ldots(p^h + \ldots + p^m)}{(p^{l+1} + p^{l+2} + \ldots + p^h)(p^{l+2} + \ldots + p^h)\ldots p^h} \quad (1)$$

**Result 2** The set of all linear distinct sub-varieties of the same dimension $h$ of $V_m$, can be identified to a system BIB design with the parameters:

$$v = \sum_{i=0}^{\frac{m}{h}} p^i, \quad b = \frac{(1 + p + p^2 + \ldots + p^m)(p + p^2 + \ldots + p^m)\ldots(p^h + \ldots + p^m)}{(1 + p + p^2 + \ldots + p^h)(p + p^2 + \ldots + p^h)\ldots p^h} \quad \text{and} \quad k = \sum_{i=0}^{\frac{m}{h}} p^i.$$  

For $l = 0$ and $l = 1$ in (1), we obtain $r$ and $\lambda$ respectively.

**Result 3** All $h$-dimensional sub-variety ($h < m$) of an $m$-dimensional projective geometry is a $h$-dimensional projective geometry. From results 1 and 2, we simply deduce Proposition 1:

**Proposition 1**

1. The set of all $m_1$-dimensional linear distinct sub-varieties ($m_1 = m-1$): $\{V(i_1): 1 \leq i_1 \leq b_1\}$ of $V_m$ is a symmetrical blocks design, said of the $1^{\text{st}}$ generation of the parameters: $(v_1, b_1, r_1, k_1, \lambda_1)$ with:

$$v_1 = b_1 = \sum_{i=0}^{m-1} p^i, \quad k_1 = r_1 = \sum_{i=0}^{m-1} p^i \quad \text{and} \quad \lambda_1 = \sum_{i=0}^{m-2} p^i.$$  

2. Residual design associates with $\{V(i_1): 1 \leq i_1 \leq b_1\}$ is a resolvable in $V_m$ of $1^{\text{st}}$ generation and denoted by $\{V^*(i_1): 1 \leq i_1 \leq b_1^*\}$. Its parameters $v_1^*, b_1^*, r_1^*, k_1^*$ et $\lambda_1^*$ are such as:

$$v_1^* = v_1 - k_1, \quad b_1^* = b_1 - 1, \quad r_1^* = r_1, \quad k_1^* = k_1 - \lambda_1, \quad \lambda_1^* = \lambda_1.$$  

**Proof:**

1. The parameters $v_1, b_1, r_1, k_1, \lambda_1$ of $\{V(i_1): 1 \leq i_1 \leq b_1\}$ deduce simply of (1) in replacing $h$ by $m_1 = m-1$ and $l = 0$ (resp. $l = 1$). Consequently:

$$v_1 = b_1 = \sum_{i=0}^{m-1} p^i, \quad k_1 = r_1 = \sum_{i=0}^{m-1} p^i \quad \text{and} \quad \lambda_1 = \sum_{i=0}^{m-2} p^i.$$  

2. We omitting any block of the symmetrical design and the treatments which are in the other blocks, The residual design obtained is resolvable. The expression of its parameters is immediate, in particular: $v_1^* = p^m$ and $b_1^* = pr_1^*$; so there is $r_1^*$ parallel design of $p$ sub-blocks each of the size $k_1^* = p^{m-1}$ containing $v_1^* = p^m$ treatments, where there is resolvability.

The method for construction of resolvable nested designs that we propose, is inspired basically from the result 3, by varying successively dimensions ($h$) of sub-varieties which we consider. More exactly, when $m_j$-dimensional
sub-varieties of the $j^{th}$ generation: $V(i_1,i_2,...,i_j)$ written in the from of $m_j = m - j : j = 1,...,n-1$ et $n < m$, then the expression of parameters of the BIB of the $j^{th}$ generation are reduced to: $b_j = 1 + p + ... + p^{m-(j-1)}$ and $r_j = k_j = \lambda_j = b_{j+1}$.

Thus, we consider each sub-variety $V(i_1)$ of the system $\{V(i_1): 1 \leq i_1 \leq b_1\}$ as a PG($m_1$, $p$), the distinct sub-varieties of the same dimension $m_2 = m - 2$. $\{V(i_1,i_2): 1 \leq i_2 \leq b_2\}$, containing in the sub-variety $V(i_1)$ could be identified to a symmetrical blocks design (said of $2^{nd}$ generation), of parameters $(v_2 = b_2, r_2 = k_2, \lambda_2)$ and so on, until stage $n (1 < n < m)$ where we obtain a symmetrical blocks design of the $n^{th}$ generation obtained by an identification to a system of $m_n$-dimensional sub-varieties ($m_n = m - n$), $\{V(i_1,...,i_{n-1}): 1 \leq i_n \leq b_n\}$ in $V(i_1,...,i_{n-1})$, of parameters $(v_n = b_n, r_n = k_n, \lambda_n)$ and which correspond to a resolvable incomplete blocks design, $\{V^*(i_1,...,i_{n-1}): 1 \leq i_n \leq b^*_n\}$ in $V(i_1,...,i_{n-1})$ referring the proposition 1.

**Theorem 1** Let a projective geometry $V_m$ defined on GF($p$) and $\{V(i_1): 1 \leq i_1 \leq b_1\}$, the system of the incomplete blocks designs of the $1^{st}$ generation.

1. For all $n = 2, ..., m-1$ : The system of the $m_n$-dimensional sub-varieties ($m_n = m - n$) $\{V(i_1,...,i_n): 1 \leq i_n \leq b_n\}$ of $n^{th}$ generation in $V(i_1,...,i_{n-1})$, is a balanced incomplete blocks design and symmetric. Its residual design is a resolvable incomplete blocks design, $\{V^*(i_1,...,i_n): 1 \leq i_n \leq b^*_n\}$ in $V(i_1,...,i_{n-1})$, of the parameters:

$$b^*_n = p + p^2 + ... + p^{m-(n-1)}, \quad r^*_n = 1 + p + p^2 + ... + p^{m-n},$$

$$k^*_n = p^{m-n} \quad \text{and} \quad \lambda^*_n = 1 + p + p^2 + ... + p^{m-(n+1)}$$

2. The union of all the resolvable incomplete blocks designs of the same generation $n$ $(1 < n < m)$ is a resolvable design noted $R_n^*(V^*_1, b^{(n)}_F, r^{(n)}_F, k^{(n)}_F, \lambda^{(n)}_F)$ of the parameters:

$$v^*_1 = p^m, \quad b^{(n)}_F = \prod_{i=1}^{i=n} b^*_i, \quad r^{(n)}_F = \prod_{i=1}^{i=n} r^*_i, \quad k^{(n)}_F = k^*_n \quad \text{and} \quad \lambda^{(n)}_F = \prod_{i=1}^{i=n} \lambda^*_i$$

**Proof**

1. Fixing the vector of coefficients $(a_{10},...,a_{1m})$. Delete the associated block with the first generation $V(i_1)$, defined by the equation: $\sum_{j=0}^{i=n} a_{ij} x_j = 0$ and also treatments which are in the other blocks of the same generation, the associated residual design is a resolvable blocks design according to the proposition 1. Moreover, the omitting of $V(i_1)$ implied the omitting of sub-varieties obtained by those of the second stage and also of the following stages. Also the omission of all the $m_j$-dimensional sub-varieties $(m_j \leq m_2)$, $j = 2,...,n$ including the equations system defined, the determined equation $V(i_1)$ and thus making residual incomplete blocks designs, the resolvable designs $\{V^*(i_1,...,i_j): 1 \leq i_j \leq b^*_j\}$ in $V(i_1,...,i_{j-1})$ for each stage $j$, always referring to Proposition 1.

2. $v^*_1 = v_1 - k_1 = p^m$, by construction. The other parameters $b^{(n)}_F, r^{(n)}_F, k^{(n)}_F$ and $\lambda^{(n)}_F$ of the $R_n^*$ design are simply deduced from the method of construction of nested sub-varieties sequentially and of the proposition 1.

**Theorem 2** For all $n = 2, ..., m-1$, it exists a resolvable design noted: $Q_n^*(v^*_1, b^*_n, r^*_n, k^*_n, \lambda^*_n)$, deduced of
the $R_n^*(v_1^n*, b_n^{(n)}, r_n^{(n)}, k_n^{(n)}, \lambda_n^{(n)})$ design, of the parameters:

$$v_1^n* = p^n, \quad b_n^{**} = \frac{b_n^{(n)}}{\alpha} r_n^{**} = \frac{r_n^{(n)}}{\alpha}, \quad k_n^{**} = k_n^* = p^{m-n}$$

and $\lambda_n^{**} = \frac{\lambda_n^{(n)}}{\alpha}$ with $\alpha = \prod_{i=1}^{i=1}N(m,m_i,m_{i+1}) = \prod_{i=1}^{i=1}(1 + p + p^2 + \ldots + p^i)$

Each $m_{j+1}$-dimensional sub-varieties, $V(i_1,i_2,...,i_{j+1})$ with $1 \leq i_{j+1} \leq b_{j+1}$ is contained in $N(m,m_j,m_{j+1}) = 1 + p + \ldots + p^j$ $m_j$-dimensional sub-varieties (refering to (1)); and therefore the total number of $m_{j+1}$-dimensional distinct sub-varieties in all $m_j$-dimensional sub-varieties is equal to: $\frac{b_j \times b_{j+1}}{N(m,m_j,m_{j+1})}$.

In the step (n), so the $m_n$-dimensional distinct sub-varieties total number is:

$$\sum_{i=1}^{i=1}b_i = \frac{\prod_{i=1}^{i=1}b_i}{\prod_{i=1}^{i=1}N(m,m_i,m_{i+1})}$$

As $b_j = 1 + p + \ldots + p^{m-(j-1)}$ and $b_j^* = p + \ldots + p^{m-(j-1)} = p(1 + \ldots + p^{m-j}) = p \cdot b_{j+1}$;

Consequently, at the stage (n) of the recursive method of construction of resolvable designs,

$$b_F^{(n)} = \prod_{i=1}^{i=1}b_i^* = p^n(1 + p + \ldots + p^{m-n}) \prod_{i=2}^{i=1}b_i = \frac{p^n + p^{n+1} + \ldots + p^m}{1 + p + \ldots + p^m} \prod_{i=1}^{i=1}b_i$$

$\prod_{i=1}^{i=1}b_i$ is divisible by $\alpha$ since $\prod_{i=1}^{i=1}b_i$ is. The same reasoning is valid for $r_n^{**}$ and $\lambda_n^{**}$.

The blocks size and the treatments number stay unchangeable.

**Theorem 3** The given projective geometry $V_m$ defined on a $GF(p)$.

1. The reduced resolvable blocks designs $Q_n^*(v_1^n*, b_n^{**}, r_n^{**}, k_n^{**}, \lambda_n^{**})$ generate a sequence of symmetrical uniform design with $v_1^n* = p^m$ runs and $r_n^{**}$ factors, each having $p^n$ levels, $n = 1, \ldots, m-1$, denoted by:

$\cup(v_1^n*; (p^n)^{**})$.

2. If $p = 2$ and $n = 1$, the uniform design associates $\cup(v^n*; 2^n)$ is identical to a Plackett and Burman design with $v^n*$ runs and $r_1^* = v^n* - 1$ factors.

**Proof**

1. The number of treatment represents the number of runs of the uniform design associated with $Q_n^*$ (cf. Fang [6,7]) and $r_n^{**}$ factors each having $\frac{b_n^{**}}{r_n^{**}} = p^n$ levels.

172
2. At step \( n = 1 \) of the recurrence and \( p = 2 \) there are: \( r_1^{**} = r_1 = \sum_{j=1}^{j=m} 2^{j-1} = 2^m - 1 = v^* - 1 \); thus by recoding the levels (1, 2) of the design \( U(v^*; 2^k) \) by (-1, +1), we find the Placket and Burman design [14] with \( v^* \) runs and with \( v^* - 1 \) factors.

Tables 1 and 2 show the different nature of obtained designs from a \( PG(m, 2) \) (resp. \( PG(m, 3) \)) for \( m = 2, 3 \) and 4.

### Table 1: \( (p = 2) \)

<table>
<thead>
<tr>
<th>Designs nature</th>
<th>Parameters</th>
<th>( m = 2 )</th>
<th>( m = 3 )</th>
<th>( m = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( BIB^{(3)} )</td>
<td>( (7,3,1) )</td>
<td>( (15,7,3) )</td>
<td>( (31,15,7) )</td>
<td></td>
</tr>
<tr>
<td>( RBIB^{(1)} )</td>
<td>( (4,6,3,2,1) )</td>
<td>( (8,14,7,4,3) )</td>
<td>( (16,30,15,8,7) )</td>
<td></td>
</tr>
<tr>
<td>( U_{v_1}^*(p^5) )</td>
<td>( U_4(2^7) )</td>
<td>( U_8(2^7) )</td>
<td>( U_{16}(2^{15}) )</td>
<td></td>
</tr>
<tr>
<td>( BIB^{(2)} )</td>
<td>#</td>
<td>( (7,3,1) )</td>
<td>( (15,7,3) )</td>
<td></td>
</tr>
<tr>
<td>( RBIB^{(2)} )</td>
<td>#</td>
<td>( (4,6,3,2,1) )</td>
<td>( (8,14,7,4,3) )</td>
<td></td>
</tr>
<tr>
<td>( U_{v_2}^*(p^5) )</td>
<td>#</td>
<td>( U_4(2^3) )</td>
<td>( U_8(2^7) )</td>
<td></td>
</tr>
<tr>
<td>( R^2(v_1^<em>, b_{2}^</em>, r_{2}^<em>, k_{2}^</em>, \lambda_{2}^*) )</td>
<td>#</td>
<td>( (8,28,7,2,1) )</td>
<td>( (16,140,35,4,7) )</td>
<td></td>
</tr>
<tr>
<td>( Q_2^* )</td>
<td>#</td>
<td>( U_8((2^2)^7) )</td>
<td>( U_{16}(2^{15}) )</td>
<td></td>
</tr>
</tbody>
</table>

### Table 2: \( (p=3) \)

<table>
<thead>
<tr>
<th>Designs nature</th>
<th>Parameters</th>
<th>( m = 2 )</th>
<th>( m = 3 )</th>
<th>( m = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( BIB^{(1)} )</td>
<td>( (13,4,1) )</td>
<td>( (40,13,4) )</td>
<td>( (121,40,13) )</td>
<td></td>
</tr>
<tr>
<td>( RBIB^{(1)} )</td>
<td>( (9,12,4,3,1) )</td>
<td>( (27,39,13,9,4) )</td>
<td>( (81,120,40,27,13) )</td>
<td></td>
</tr>
<tr>
<td>( U_{v_1}^*(p^5) )</td>
<td>( U_9(3^7) )</td>
<td>( U_{15}(3^{13}) )</td>
<td>( U_{41}(3^{40}) )</td>
<td></td>
</tr>
<tr>
<td>( BIB^{(2)} )</td>
<td>#</td>
<td>( (13,4,1) )</td>
<td>( (40,13,4) )</td>
<td></td>
</tr>
<tr>
<td>( RBIB^{(2)} )</td>
<td>#</td>
<td>( (9,12,4,3,1) )</td>
<td>( (27,39,13,9,4) )</td>
<td></td>
</tr>
<tr>
<td>( U_{v_2}^*(p^5) )</td>
<td>#</td>
<td>( U_9(3^4) )</td>
<td>( U_{27}(3^{13}) )</td>
<td></td>
</tr>
<tr>
<td>( R^2(v_1^<em>, b_{2}^</em>, r_{2}^<em>, k_{2}^</em>, \lambda_{2}^*) )</td>
<td>#</td>
<td>( (27,468,3,52,4) )</td>
<td>( (81,4680,520,9,52) )</td>
<td></td>
</tr>
<tr>
<td>( Q_2^* )</td>
<td>#</td>
<td>( (27,117,13,3,1) )</td>
<td>( (81,1170,130,9,13) )</td>
<td></td>
</tr>
</tbody>
</table>
To facilitate the use of those designs, we propose the package *R PGM 2* that provides the configuration of (BIB, Resolvable and Uniform) designs, for \( p = 2 \).

Example 2 below (resp. example 3), illustrates the designs of 1\(^{st}\) generation as represented in Figure 1 and Figure 2 (resp. designs of 2\(^{nd}\) generation, the last stage of our method of construction) and according to the outputs of the package *R PGM 2*.

**Example 2**

```r
> Bie(3)$BIB # BIB(15,7,3)

```
Example 3

#RBIB(8,28,7,2,1)

```
[,1] [,2]
[1,] 1   9
[2,] 5   13
[3,] 1   5
[4,] 9   13
[5,] 1   13
[6,] 5   9
[7,] 3   11
[8,] 7   15
[9,] 3   7
[10,] 11  15
```
REFERENCES


