ON SOME DIFFERENTIAL SANDWICH THEOREMS USING SĂLĂGEAN OPERATOR AND RUSCHEWEYH OPERATOR

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ABSTRACT
In this work we define a new operator using the Sălăgean operator and Ruscheweyh operator. Denote by \( SR_{m,n} \) the Hadamard product of the Sălăgean operator \( S^m \) and Ruscheweyh operator \( R^n \), given by \( SR_{m,n} : \mathcal{A} \rightarrow \mathcal{A} \), \( SR_{m,n} f(z) = (S^m * R^n) f(z) \) and \( \mathcal{A}_n = \{ f \in \mathcal{H}(U) : f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \ldots, z \in U \} \) is the class of normalized analytic functions with \( \mathcal{A}_1 = \mathcal{A} \). The purpose of this paper is to introduce sufficient conditions for subordination and superordination involving the operator \( SR_{m,n} \) and also to obtain sandwich-type results.

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1. INTRODUCTION
Let \( \mathcal{H}(U) \) be the class of analytic function in the open unit disc of the complex plane \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( \mathcal{H}(a,n) \) be the subclass of \( \mathcal{H}(U) \) consisting of functions of the form \( f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \). Let \( \mathcal{A}_n = \{ f \in \mathcal{H}(U) : f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \ldots, z \in U \} \) and \( \mathcal{A} = \mathcal{A}_1 \).

Denote by \( K = \left\{ f \in \mathcal{A} : \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) + 1 > 0, z \in U \right\} \), the class of normalized convex functions in \( U \).

Let the functions \( f \) and \( g \) be analytic in \( U \). We say that the function \( f \) is subordinate to \( g \), written \( f \prec g \), if there exists a Schwarz function \( w \), analytic in \( U \), with \( w(0) = 0 \) and \( |w(z)| < 1 \), for all \( z \in U \), such that \( f(z) = g(w(z)) \), for all \( z \in U \). In particular, if the function \( g \) is univalent in \( U \), the above subordination is equivalent to \( f(0) = g(0) \) and \( f(U) \subset g(U) \).

Let \( \psi : \mathbb{C}^2 \times U \rightarrow \mathbb{C} \) and \( h \) be an univalent function in \( U \). If \( p \) is analytic in \( U \) and satisfies the second order differential subordination
\[
\psi(p(z),zp'(z),z^2 p''(z);z) \prec h(z), \quad \text{for } z \in U,
\]
then \( p \) is called a solution of the differential subordination. The univalent function \( q \) is called a dominant of the solutions of the differential subordination, or more simply a dominant, if \( p \prec q \) for all \( p \) satisfying (1). A dominant \( q \) that satisfies \( q \prec q \) for all dominants \( q \) of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of \( U \).

Let \( \psi : \mathbb{C}^2 \times U \rightarrow \mathbb{C} \) and \( h \) analytic in \( U \). If \( p \) and \( \psi(p(z),zp'(z),z^2 p''(z);z) \) are univalent and if \( p \) satisfies the second order differential superordination
\[
h(z) \prec \psi(p(z),zp'(z),z^2 p''(z);z), \quad z \in U,
\]
then \( p \) is a solution of the differential superordination (2) (if \( f \) is subordinate to \( F \), then \( F \) is called to be superordinate to \( f \)). An analytic function \( q \) is called a subordinant if \( q \prec p \) for all \( p \) satisfying (2). An
univalent subordinant \( \tilde{q} \) that satisfies \( q \prec \tilde{q} \) for all subordinants \( q \) of (2) is said to be the best subordinant.

Miller and Mocanu [8] obtained conditions \( h, q \) and \( \psi \) for which the following implication holds

\[
h(z) \prec \psi(p(z), zp'(z), z^2 p''(z), z) \Rightarrow q(z) \prec p(z)
\]

For two functions \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \) and \( g(z) = z + \sum_{j=2}^{\infty} b_j z^j \) analytic in the open unit disc \( U \), the Hadamard product (or convolution product) of \( f(z) \) and \( g(z) \), written as \( (f \ast g)(z) \), is defined by

\[
f(z) \ast g(z) = (f \ast g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j.
\]

**Definition 1.1** (Sălăgean [11]) For \( f \in \mathcal{A} \), and \( n \in \mathbb{N} \), the operator \( S^n \) is defined by \( S^n : \mathcal{A} \to \mathcal{A} \),

\[
S^0 f(z) = f(z),
\]

\[
S^1 f(z) = zf'(z),
\]

\[
\ldots
\]

\[
S^{n+1} f(z) = z \left(S^n f(z)\right), \quad z \in U.
\]

**Remark 1.1** If \( f \in \mathcal{A} \), \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \), then \( S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \), \( z \in U \).

**Definition 1.2** (Ruscheweyh [10]) For \( f \in \mathcal{A} \) and \( n \in \mathbb{N} \), the operator \( R^n \) is defined by \( R^n : \mathcal{A} \to \mathcal{A} \),

\[
R^0 f(z) = f(z),
\]

\[
R^1 f(z) = zf'(z),
\]

\[
\ldots
\]

\[
(n+1)R^{n+1} f(z) = z \left(R^n f(z)\right) + nR^n f(z), \quad z \in U.
\]

**Remark 1.2** If \( f \in \mathcal{A} \), \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \), then \( R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j \) for \( z \in U \).

**Definition 1.3** ([7]) Let \( n, m \in \mathbb{N} \). Denote by \( SR^{m,n} : \mathcal{A} \to \mathcal{A} \) the operator given by the Hadamard product of the generalized Sălăgean operator \( D^n \) and the Ruscheweyh operator \( R^n \),

\[
SR^{m,n} f(z) = \left(S^m \ast R^n\right)f(z), \quad (3)
\]

for any \( z \in U \) and each nonnegative integers \( m, n \).

**Remark 1.3** If \( f \in \mathcal{A} \) and \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \), then \( SR^{m,n} f(z) = z + \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j \), \( z \in U \).

**Remark 1.4** For \( m = n \), we obtain the Hadamard product \( SR^n \) [1] of the Sălăgean operator \( S^n \) and Ruscheweyh derivative \( R^n \), which was studied in [2], [3].

Using simple computation one obtains the next result.
Proposition 1.1 ([7]) For $m,n \in \mathbb{N}$ we have

$$SR^{m+1,n} f(z) = z(SR^{m,n} f(z))$$

and

$$z(SR^{m,n} f(z)) = (n+1)SR^{m,n+1} f(z) - nSR^{m,n} f(z).$$

The purpose of this paper is to derive the several subordination and superordination results involving a differential operator. Furthermore, we studied the results of M. Darus, K. Al-Shaq [6], Shanmugam, Ramachandran, Darus and Sivasubramanian [12].

In order to prove our subordination and superordination results, we make use of the following known results.

Definition 1.4 [9] Denote by $Q$ the set of all functions $f$ that are analytic and injective on $\overline{U} \setminus E(f)$, where $E(f) = \{\xi \in \partial U : \lim_{z \to \xi} f(z) = \infty\}$, and are such that $f^{'}(\xi) \neq 0$ for $\xi \in \partial U \setminus E(f)$.

Lemma 1.1 [9] Let the function $q$ be univalent in the unit disc $U$ and $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq^{'}(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$.

Suppose that

1. $Q$ is starlike univalent in $U$ and

2. $Re \left( \frac{zh^{'}(z)}{Q(z)} \right) > 0$ for $z \in U$.

If $p$ is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp^{'}(z)\phi(p(z)) < \theta(q(z)) + zq^{'}(z)\phi(q(z)),$$

then $p(z) \prec q(z)$ and $q$ is the best dominant.

Lemma 1.2 [5] Let the function $q$ be convex univalent in the open unit disc $U$ and $\nu$ and $\phi$ be analytic in a domain $D$ containing $q(U)$. Suppose that

1. $Re \left( \frac{\nu(q(z))}{\phi(q(z))} \right) > 0$ for $z \in U$ and

2. $\psi(z) = zq^{'}(z)\phi(q(z))$ is starlike univalent in $U$.

If $p(z) \in \mathcal{H}[q(0),1] \cap Q$, with $p(U) \subseteq D$ and $\nu(p(z)) + zp^{'}(z)\phi(p(z))$ is univalent in $U$ and

$$\nu(q(z)) + zq^{'}(z)\phi(q(z)) \prec \nu(p(z)) + zp^{'}(z)\phi(p(z)),$$

then $q(z) \prec p(z)$ and $q$ is the best subordinant.

2. MAIN RESULTS

Considering $\lambda = 1$ in [4] we obtain the following results.

Theorem 2.1 Let \(\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)} \in \mathcal{H}(U)\), $z \in U$, $f \in \mathcal{A}$, $m,n \in \mathbb{N}$, $\lambda \geq 0$ and let the function $q(z)$ be convex and univalent in $U$ such that $q(0) = 1$. Assume that
\[
\text{Re} \left( 1 + \frac{\alpha}{\mu} + 2\beta \frac{q(z)}{\mu} + \frac{zq^{''}(z)}{q(z)} \right) > 0, \quad z \in U, \tag{6}
\]

for \(\alpha, \beta, \mu \in \mathbb{C}, \mu \neq 0, \quad z \in U,\) and

\[
\psi_{\lambda, m}^{n, n}(\alpha, \beta, \mu, z) := (-n\mu + \alpha)\frac{SR_{m+1, n} f(z)}{SR_{m, n} f(z)} + \mu(n+1)^{2} \frac{SR_{m+1, n} f(z)}{SR_{m, n} f(z)} + \mu(n+1)(n+2) \frac{SR_{m+2, n} f(z)}{SR_{m, n} f(z)} + (\beta - \mu) \left( \frac{SR_{m+1, n} f(z)}{SR_{m, n} f(z)} \right)^{2}. \tag{7}
\]

If \(q\) satisfies the following subordination

\[
\psi_{\lambda, m}^{n, n}(\alpha, \beta, \mu, z) \prec \alpha q(z) + \beta (q(z))^2 + \mu \varepsilon q(z), \tag{8}
\]

for \(\alpha, \beta, \mu \in \mathbb{C}, \mu \neq 0\) then

\[
\frac{SR_{m+1, n} f(z)}{SR_{m, n} f(z)} \prec q(z), \quad z \in U, \tag{9}
\]

and \(q\) is the best dominant.

**Proof.** Let the function \(p\) be defined by \(p(z) := \frac{SR_{m+1, n} f(z)}{SR_{m, n} f(z)}, \quad z \in U, \quad z \neq 0, \quad f \in A.\) The function \(p\) is analytic in \(U\) and \(p(0) = 1.\)

Differentiating this function, with respect to \(z,\) we get

\[
zp^{'}(z) = \frac{z(SR_{m+1, n} f(z))'}{SR_{m, n} f(z)} - \frac{SR_{m+1, n} f(z)}{SR_{m, n} f(z)} \frac{z(SR_{m, n} f(z))'}{SR_{m, n} f(z)} +
\]

\[
(n+1)(n+2) \frac{SR_{m+2, n} f(z)}{SR_{m, n} f(z)} - \left( \frac{SR_{m+1, n} f(z)}{SR_{m, n} f(z)} \right)^{2} +
\]

\[
(n+1)(n+2) \frac{SR_{m+2, n} f(z)}{SR_{m, n} f(z)} - \left( \frac{SR_{m+1, n} f(z)}{SR_{m, n} f(z)} \right)^{2}. \tag{10}
\]

By setting \(\theta(w) := \alpha w + \beta w^2\) and \(\phi(w) := \mu, \quad \alpha, \beta, \mu \in \mathbb{C}, \mu \neq 0\) it can be easily verified that \(\theta\) is analytic in \(\mathbb{C},\phi\) is analytic in \(\mathbb{C} \setminus \{0\}\) and that \(\phi(w) \neq 0, \quad w \in \mathbb{C} \setminus \{0\}.\)

Also, by letting \(Q(z) = zq(z)\phi(q(z)) = \mu \varepsilon q(z),\) we find that \(Q(z)\) is starlike univalent in \(U.\)

Let \(h(z) = \theta(q(z)) + \phi(q(z)) = \alpha q(z) + \beta (q(z))^2 + \mu \varepsilon q(z), \quad z \in U.\)

If we derive the function \(Q,\) with respect to \(z,\) perform calculations, we have

\[
\text{Re} \left( \frac{zh'(z)}{Q(z)} \right) = \text{Re} \left( 1 + \frac{\alpha}{\mu} + 2\beta \frac{q(z)}{\mu} + \frac{zq^{''}(z)}{q(z)} \right) > 0.
\]
By using (10), we obtain
\[ a \phi(z) + \beta(p(z))^2 + \mu \psi(z) = \]
\[ (-n\mu + \alpha)\frac{SR_{m+1,n}^n f(z)}{SR_{m,n}^n f(z)} - \mu(n+1)^2 \frac{SR_{m+1,n}^n f(z)}{SR_{m,n}^n f(z)} + \mu(n+1)(n+2) \frac{SR_{m,n+2}^n f(z)}{SR_{m,n}^n f(z)} + (\beta - \mu) \left( \frac{SR_{m+1,n}^n f(z)}{DS_{m,n}^n f(z)} \right)^2. \]

By using (8), we have
\[ a \phi(z) + \beta(p(z))^2 + \mu \psi(z) \prec a \phi(z) + \beta(q(z))^2 + \mu \psi(z). \]

Therefore, the conditions of Lemma 1.1 are met, so we have
\[ p(z) \prec q(z), \quad z \in U, \text{ i.e. } \frac{SR_{m+1,n}^n f(z)}{SR_{m,n}^n f(z)} \prec q(z), \]
\[ z \in U, \text{ and } q \text{ is the best dominant.} \]

**Corollary 2.2** Let \( q(z) = \frac{1 + Az}{1 + Bz} \), \(-1 \leq B < A \leq 1\), \( m, n \in \mathbb{N}, \lambda \geq 0, \quad z \in U. \) Assume that (6) holds. If \( f \in \mathcal{A} \) and
\[ \psi_{\lambda}^{m,n}(\alpha, \beta, \mu; z) \prec a \alpha \frac{1 + Az}{1 + Bz} + \beta \left( \frac{1 + Az}{1 + Bz} \right)^2 + \mu \left( A - B \right) z \left( \frac{1 + Az}{1 + Bz} \right)^2, \]
for \( \alpha, \beta, \mu \in \mathbb{C}, \mu \neq 0, \ -1 \leq B < A \leq 1, \) where \( \psi_{\lambda}^{m,n} \) is defined in (7), then
\[ \frac{SR_{m+1,n}^n f(z)}{SR_{m,n}^n f(z)} \prec \frac{1 + Az}{1 + Bz}, \]
and \( \frac{1 + Az}{1 + Bz} \) is the best dominant.

**Proof.** For \( q(z) = \frac{1 + Az}{1 + Bz} \), \(-1 \leq B < A \leq 1, \) in Theorem 2.1 we get the corollary.

**Corollary 2.3** Let \( q(z) = \left( \frac{1 + z}{1 - z} \right)^\gamma, m, n \in \mathbb{N}, \lambda \geq 0, z \in U. \) Assume that (6) holds. If \( f \in \mathcal{A} \) and
\[ \psi_{\lambda}^{m,n}(\alpha, \beta, \mu; z) \prec a \alpha \left( \frac{1 + z}{1 - z} \right)^\gamma + \beta \left( \frac{1 + z}{1 - z} \right)^{2\gamma} + \mu \left( \frac{2\gamma z}{1 - z^2} \right) \left( \frac{1 + z}{1 - z} \right)^{\gamma - 1}, \]
for \( \alpha, \mu \in \mathbb{C}, \ 0 < \gamma \leq 1, \mu \neq 0, \) where \( \psi_{\lambda}^{m,n} \) is defined in (7), then
\[ \frac{SR_{m+1,n}^n f(z)}{SR_{m,n}^n f(z)} \prec \left( \frac{1 + z}{1 - z} \right)^\gamma, \]
and \( \left( \frac{1 + z}{1 - z} \right)^\gamma \) is the best dominant.

**Proof.** Corollary follows by using Theorem 2.1 for \( q(z) = \left( \frac{1 + z}{1 - z} \right)^\gamma, \ 0 < \gamma \leq 1. \)

**Theorem 2.4** Let \( q \) be convex and univalent in \( U, \) such that \( q(0) = 1, \) \( m, n \in \mathbb{N}, \lambda \geq 0. \) Assume that
\[
\Re\left(\frac{q'(z)}{\mu}(\alpha + 2\beta q(z))\right) > 0, \text{ for } \alpha, \mu, \beta \in \mathbb{C}, \mu \neq 0, z \in U. \quad (11)
\]

If \( f \in \mathcal{A}, \quad \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)} \in \mathcal{H}[q(0),1] \cap Q \) and \( \psi_{\lambda}^{m,n}(\alpha, \beta, \mu; z) \) is univalent in \( U \), where \( \psi_{\lambda}^{m,n}(\alpha, \beta, \mu; z) \) is as defined in (7), then
\[
\alpha q(z) + \beta(q(z))^2 + \mu \varepsilon q(z) \prec \psi_{\lambda}^{m,n}(\alpha, \beta, \mu; z), \quad z \in U, \quad (12)
\]

implies
\[
q(z) \prec \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}, \quad z \in U, \quad (13)
\]

and \( q \) is the best subordinate.

**Proof.** Let the function \( p \) be defined by \( p(z) := \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}, \quad z \in U, \quad z \neq 0, \quad f \in \mathcal{A}. \)

By setting \( v(w) := \alpha v + \beta w^2 \) and \( \phi(w) := \mu \) it can be easily verified that \( v \) is analytic in \( \mathbb{C} \), \( \phi \) is analytic in \( \mathbb{C} \setminus \{0\} \) and that \( \phi(w) \neq 0, \quad w \in \mathbb{C} \setminus \{0\} \).

Since
\[
\frac{v'(q(z))}{\phi(q(z))} = \frac{q'(z)}{\mu}(\alpha + 2\beta q(z)),
\]

it follows that
\[
\Re\left(\frac{v'(q(z))}{\phi(q(z))}\right) = \Re\left(\frac{q'(z)}{\mu}(\alpha + 2\beta q(z))\right) > 0, \quad \text{for } \mu, \xi, \beta \in \mathbb{C}, \quad \mu \neq 0.
\]

By using (12) we obtain
\[
\alpha q(z) + \beta(q(z))^2 + \mu \varepsilon q(z) \prec \alpha q(z) + \beta(q(z))^2 + \mu \dot{q}(z).
\]

Using Lemma 1.2, we have
\[
q(z) \prec p(z) = \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}, \quad z \in U,
\]

and \( q \) is the best subordinate.

**Corollary 2.5** Let \( q(z) = \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad m, n \in \mathbb{N}, \quad \lambda \geq 0. \) Assume that (11) holds. If \( f \in \mathcal{A}, \)
\[
\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)} \in \mathcal{H}[q(0),1] \cap Q \quad \text{and}
\]

\[
\alpha \frac{1 + Az}{1 + Bz} + \beta \left(\frac{1 + Az}{1 + Bz}\right)^2 + \mu \frac{(A - B)z}{(1 + Bz)} \prec \psi_{\lambda}^{m,n}(\alpha, \beta, \mu; z),
\]

for \( \alpha, \mu, \beta \in \mathbb{C}, \quad \mu \neq 0, \quad -1 \leq B < A \leq 1, \) where \( \psi_{\lambda}^{m,n} \) is defined in (7), then
\[
\frac{1 + Az}{1 + Bz} \prec \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}
\]

and \( \frac{1 + Az}{1 + Bz} \) is the best subordinate.
Proof. For $q(z) = \frac{1 + A_z}{1 + B_z}$, $-1 \leq B < A \leq 1$ in Theorem 2.4 we get the corollary.

Corollary 2.6 Let $q(z) = \left(\frac{1 + z}{1 - z}\right)^\gamma$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (11) holds. If $f \in \mathcal{A}$, $\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and

$$\alpha \left(\frac{1 + z}{1 - z}\right)^\gamma + \beta \left(\frac{1 + z}{1 - z}\right)^{2\gamma} + \mu \frac{2\gamma z}{1 - z} \left(\frac{1 + z}{1 - z}\right)^{\gamma - 1} < \psi_{m,n}^\alpha (\alpha, \beta, \mu; z),$$

for $\alpha, \mu, \beta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\mu \neq 0$, where $\psi_{m,n}^\alpha$ is defined in (7), then

$$\left(\frac{1 + z}{1 - z}\right)^\gamma < \frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)}$$

and $\left(\frac{1 + z}{1 - z}\right)^\gamma$ is the best subordinant.

Proof. Corollary follows by using Theorem 2.4 for $q(z) = \left(\frac{1 + z}{1 - z}\right)^\gamma$, $0 < \gamma \leq 1$.

Combining Theorem 2.1 and Theorem 2.4, we state the following sandwich theorem.

Theorem 2.7 Let $q_1$ and $q_2$ be analytic and univalent in $U$ such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, for all $z \in U$, with $zq_1(z)$ and $zq_2(z)$ being starlike univalent. Suppose that $q_1$ satisfies (6) and $q_2$ satisfies (11). If $f \in \mathcal{A}$, $\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and $\psi_{m,n}^\alpha (\alpha, \beta, \mu; z)$ is as defined in (7) univalent in $U$, then

$$\alpha q_1(z) + \beta q_2(z) > \psi_{m,n}^\alpha (\alpha, \beta, \mu; z) > \alpha q_1(z) + \beta q_2(z),$$

for $\alpha, \mu, \beta \in \mathbb{C}$, $\mu \neq 0$, implies

$$q_1(z) < \frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)} < q_2(z), \quad \delta \in \mathbb{C}, \delta \neq 0,$$

and $q_1$ and $q_2$ are respectively the best subordinant and the best dominant.

For $q_1(z) = \frac{1 + A_2z}{1 + B_2z}$, $q_2(z) = \frac{1 + A_2z}{1 + B_2z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.8 Let $m, n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (6) and (11) hold for $q_1(z) = \frac{1 + A_2z}{1 + B_2z}$ and $q_2(z) = \frac{1 + A_2z}{1 + B_2z}$, respectively. If $f \in \mathcal{A}$, $\frac{SR^{m+1,n}f(z)}{SR^{m,n}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and

$$\alpha \frac{1 + A_2z}{1 + B_2z} + \beta \left(\frac{1 + A_2z}{1 + B_2z}\right)^2 + \mu \frac{(A_1 - B_1)z}{(1 + B_2z)^2} < \psi_{m,n}^\alpha (\alpha, \beta, \mu; z)$$

for $\alpha, \mu, \beta \in \mathbb{C}$, $\mu \neq 0$,
\[ \alpha \frac{1 + A_z z}{1 + B_z z} + \beta \left( \frac{1 + A_z z}{1 + B_z z} \right)^2 + \mu \frac{(A_z - B_z) z}{(1 + B_z z)^2}, \]

for \( \alpha, \mu, \beta \in \mathbb{C}, \mu \neq 0, \ -1 \leq B_z \leq B_1 < A_z \leq \lambda \leq 1, \) where \( \psi_{\lambda}^{m,n} \) is defined in (7), then

\[ \frac{1 + A_z z}{1 + B_z z} < \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} < \frac{1 + A_z z}{1 + B_z z}, \]

hence \( \frac{1 + A_z z}{1 + B_z z} \) and \( \frac{1 + A_z z}{1 + B_z z} \) are the best subordinant and the best dominant, respectively.

**Theorem 2.9** Let \( \left( \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} \right)^{\delta} \in \mathcal{H}(U), f \in \mathcal{A}, z \in U, \delta \in \mathbb{C}, \delta \neq 0, m,n \in \mathbb{N}, \lambda \geq 0 \) and let the function \( q(z) \) be convex and univalent in \( U \) such that \( q(0) = 1, z \in U. \) Assume that

\[ \text{Re} \left( \frac{\alpha + \beta}{\beta} + \frac{z q(z)}{q(z)} \right) > 0, \]  

(14)

for \( \alpha, \beta \in \mathbb{C}, \beta \neq 0, z \in U, \) and

\[ \psi_{\lambda}^{m,n}(\alpha, \beta; z) := \left( \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} \right)^{\delta} (\alpha - n \delta \beta - \delta \beta (n+1)^2 \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} + \delta \beta (n+1)(n+2) \frac{\text{SR}^{m+2,n} f(z)}{\text{SR}^{m+1,n} f(z)} - \delta \beta \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} ) \]  

(15)

If \( q \) satisfies the following subordination

\[ \psi_{\lambda}^{m,n}(\alpha, \beta; z) < \alpha q(z) + \beta z q(z), \]

(16)

for \( \alpha, \beta \in \mathbb{C}, \beta \neq 0, z \in U, \) then

\[ \left( \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} \right)^{\delta} < q(z), z \in U, \delta \in \mathbb{C}, \delta \neq 0, \]

(17)

and \( q \) is the best dominant.

**Proof.** Let the function \( p \) be defined by \( p(z) := \left( \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} \right)^{\delta}, z \in U, z \neq 0, f \in \mathcal{A}. \) The function \( p \) is analytic in \( U \) and \( p(0) = 1. \)

We have \( z p^\prime(z) = \delta \left( \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} \right)^{\delta} \frac{\text{SR}^{m,n} f(z)}{\text{SR}^{m,n} f(z)} \left( \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} \right)^{\delta} = \]

\[ \delta \left( \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} \right)^{\delta} \frac{\text{SR}^{m,n} f(z)}{\text{SR}^{m,n} f(z)} \left( \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} \right)^{\delta} \frac{\text{SR}^{m,n} f(z)}{\text{SR}^{m,n} f(z)} \left( \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} \right)^{\delta} \]

By using the identity (4) and (5), we obtain

\[ z p^\prime(z) = \delta \left( \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} \right)^{\delta} \frac{\text{SR}^{m,n} f(z)}{\text{SR}^{m,n} f(z)} \left( \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} \right)^{\delta} \frac{\text{SR}^{m,n} f(z)}{\text{SR}^{m,n} f(z)} \left( \frac{\text{SR}^{m+1,n} f(z)}{\text{SR}^{m,n} f(z)} \right)^{\delta} \]
\[(n+1)^2 \left\{ \frac{SR^{m,n+1} f(z)}{SR^{m,n} f(z)} + (n+1)(n+2) \frac{SR^{m,n+2} f(z)}{SR^{m,n} f(z)} - \left( \frac{SR^{m,n+1} f(z)}{SR^{m,n} f(z)} \right)^2 \right\} - \frac{SR^{m,n+1} f(z)}{SR^{m,n} f(z)} \right\} \]  

so, we obtain

\[zp'(z) = \delta \left( \frac{SR^{m,n+1} f(z)}{SR^{m,n} f(z)} \right) \left[ -n - \right] \]

\[(n+1)^2 \left\{ \frac{SR^{m,n+1} f(z)}{SR^{m,n+1} f(z)} + (n+1)(n+2) \frac{SR^{m,n+2} f(z)}{SR^{m,n} f(z)} - \frac{SR^{m,n+1} f(z)}{SR^{m,n} f(z)} \right\} \]  

By setting \( \theta(w) := \alpha w \) and \( \phi(w) := \beta \), it can be easily verified that \( \theta \) is analytic in \( \mathbb{C} \), \( \phi \) is analytic in \( \mathbb{C} \setminus \{0\} \) and that \( \phi(w) \neq 0, \ w \in \mathbb{C} \setminus \{0\} \).

Also, by letting \( Q(z) = \frac{zq}{\phi(q(z))} = \beta q(z) \), we find that \( Q(z) \) is starlike univalent in \( U \).

Let \( h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \beta q(z) \).

We have \( \text{Re} \left( \frac{zh(z)}{Q(z)} \right) = \text{Re} \left( \frac{\alpha + \beta}{\beta} + \frac{zq(z)}{q(z)} \right) > 0 \).

By using (19), we obtain \( \alpha p(z) + \beta z p'(z) = \left( \frac{SR^{m,n+1} f(z)}{SR^{m,n} f(z)} \right) \left[ \alpha - n \beta^2 - \right] \)

\[\delta \beta (n+1)^2 \left( \frac{SR^{m,n+1} f(z)}{SR^{m,n+1} f(z)} + \delta \beta (n+1)(n+2) \frac{SR^{m,n+2} f(z)}{SR^{m,n} f(z)} - \delta \beta \frac{SR^{m,n+1} f(z)}{SR^{m,n} f(z)} \right) \]

By using (16), we have \( \alpha p(z) + \beta z p'(z) < \alpha q(z) + \beta q(z) \).

From Lemma 1.1, we have \( p(z) < q(z) \), \( z \in U \), i.e. \( \left( \frac{SR^{m,n+1} f(z)}{SR^{m,n} f(z)} \right)^{\delta} < q(z) \), \( z \in U, \delta \in \mathbb{C}, \delta \neq 0 \) and \( q \) is the best dominant.

**Corollary 2.10** Let \( q(z) = \frac{1 + Az}{1 + Bz} \), \( z \in U, \ -1 \leq B < A \leq 1, \ m, n \in \mathbb{N}, \lambda \geq 0 \). Assume that (14) holds. If \( f \in \mathcal{A} \) and

\[\psi^{m,n}_\lambda(\alpha, \beta; z) < \alpha \frac{1 + Az}{1 + Bz} + \beta \frac{(A-B)z}{(1+Bz)^2}, \]

for \( \alpha, \beta \in \mathbb{C}, \beta \neq 0, \ -1 \leq B < A \leq 1 \), where \( \psi^{m,n}_\lambda \) is defined in (15), then

\[\left( \frac{SR^{m,n+1} f(z)}{SR^{m,n} f(z)} \right)^{\delta} < \frac{1 + Az}{1 + Bz}, \ \delta \in \mathbb{C}, \delta \neq 0, \]

and \( \frac{1 + Az}{1 + Bz} \) is the best dominant.
Proof. For \( q(z) = \frac{1 + Az}{1 + Bz} \), \(-1 \leq B < A \leq 1\), in Theorem 2.9 we get the corollary.

**Corollary 2.11** Let \( q(z) = \left(\frac{1 + z}{1 - z}\right)^\gamma \), \( m, n \in \mathbb{N}, \ \lambda \geq 0 \). Assume that (14) holds. If \( f \in \mathcal{A} \) and

\[
\psi_{\lambda}^{m,n}(\alpha, \beta, \mu; z) < \alpha \left(\frac{1 + z}{1 - z}\right)^\gamma + \beta \frac{2\gamma z}{1 - z^2} \left(\frac{1 + z}{1 - z}\right)^{\gamma - 1},
\]

for \( \alpha, \beta \in \mathbb{C}, \ 0 < \gamma \leq 1, \ \beta \neq 0 \), where \( \psi_{\lambda}^{m,n} \) is defined in (15), then

\[
\left(\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}\right)^\delta < \left(\frac{1 + z}{1 - z}\right)^\gamma, \ \delta \in \mathbb{C}, \delta \neq 0,
\]

and \( \left(\frac{1 + z}{1 - z}\right)^\gamma \) is the best dominant.

**Proof.** Corollary follows by using Theorem 2.9 for \( q(z) = \left(\frac{1 + z}{1 - z}\right)^\gamma, \ 0 < \gamma \leq 1 \).

**Theorem 2.12** Let \( q \) be convex and univalent in \( U \) such that \( q(0) = 1 \). Assume that

\[\text{Re}\left(\frac{\alpha}{\beta} q'(z)\right) > 0, \text{ for } \alpha, \beta \in \mathbb{C}, \beta \neq 0. \tag{20}\]

If \( f \in \mathcal{A}, \left(\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}\right)^\delta \in \mathcal{H}[q(0),1] \cap Q \) and \( \psi_{\lambda}^{m,n}(\alpha, \beta; z) \) is univalent in \( U \), where \( \psi_{\lambda}^{m,n}(\alpha, \beta; z) \) is as defined in (15), then

\[\alpha q(z) + \beta z q'(z) < \psi_{\lambda}^{m,n}(\alpha, \beta; z) \tag{21}\]

implies

\[q(z) < \left(\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}\right)^\delta, \ \delta \in \mathbb{C}, \delta \neq 0, \ z \in U, \tag{22}\]

and \( q \) is the best subordinant.

**Proof.** Let the function \( p \) be defined by \( p(z) := \left(\frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)}\right)^\delta, \ z \in U, \ z \neq 0, \ \delta \in \mathbb{C}, \delta \neq 0, \ f \in \mathcal{A} \).

The function \( p \) is analytic in \( U \) and \( p(0) = 1 \).

By setting \( \nu(w) := \alpha w \) and \( \phi(w) := \beta \) it can be easily verified that \( \nu \) is analytic in \( \mathbb{C} \), \( \phi \) is analytic in \( \mathbb{C} \setminus \{0\} \) and that \( \phi(w) \neq 0, \ w \in \mathbb{C} \setminus \{0\} \).

Since \( \frac{\nu'(q(z))}{\phi(q(z))} = \frac{\alpha}{\beta} q'(z) \), it follows that \( \text{Re}\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = \text{Re}\left(\frac{\alpha}{\beta} q'(z)\right) > 0, \) for \( \alpha, \beta \in \mathbb{C}, \beta \neq 0 \).

Now, by using (21) we obtain
\[ \alpha q(z) + \beta q'(z) \prec \alpha q(z) + \beta q'(z), \quad z \in U. \]

From Lemma 1.2, we have
\[ q(z) \prec p(z) = \left( \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)} \right)^\delta, \quad z \in U, \delta \in \mathbb{C}, \delta \neq 0, \]
and \( q \) is the best subordinate.

**Corollary 2.13** Let \( q(z) = \frac{1 + A z}{1 + B z}, \quad -1 \leq B < A \leq 1, \quad z \in U, \ m, n \in \mathbb{N}, \ \lambda \geq 0. \) Assume that (20) holds. If
\[ f \in \mathcal{A}, \left( \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)} \right)^\delta \in \mathcal{H}[q(0),1] \cap Q, \ \delta \in \mathbb{C}, \ \delta \neq 0 \]
and
\[ \alpha \frac{1 + A z}{1 + B z} + \beta \frac{(A-B)z}{(1+Bz)^2} \prec \psi^m_n(\alpha, \beta; z), \]
for \( \alpha, \beta \in \mathbb{C}, \ \beta \neq 0, \ -1 \leq B < A \leq 1, \) where \( \psi^m_n \) is defined in (15), then
\[ \frac{1 + A z}{1 + B z} \prec \left( \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)} \right)^\delta, \delta \in \mathbb{C}, \delta \neq 0, \]
and \( \frac{1 + A z}{1 + B z} \) is the best subordinate.

**Proof.** For \( q(z) = \frac{1 + A z}{1 + B z}, \quad -1 \leq B < A \leq 1, \) in Theorem 2.12 we get the corollary.

**Corollary 2.14** Let \( q(z) = \left( \frac{1 + z}{1 - z} \right)^\gamma, \ m, n \in \mathbb{N}, \ \lambda \geq 0. \) Assume that (20) holds. If \( f \in \mathcal{A}, \left( \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)} \right)^\delta \in \mathcal{H}[q(0),1] \cap Q \) and
\[ \alpha \left( \frac{1 + z}{1 - z} \right)^\gamma + \beta \frac{2z}{1 - z^2} \left( \frac{1 + z}{1 - z} \right)^{\gamma-1} \prec \psi^m_n(\alpha, \beta, \mu; z), \]
for \( \alpha, \beta \in \mathbb{C}, \ \beta \neq 0, 
0 < \gamma \leq 1, \) where \( \psi^m_n \) is defined in (15), then
\[ \left( \frac{1 + z}{1 - z} \right)^\gamma \prec \left( \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)} \right)^\delta, \delta \in \mathbb{C}, \delta \neq 0, \]
and \( \left( \frac{1 + z}{1 - z} \right)^\gamma \) is the best subordinate.

**Proof.** Corollary follows by using Theorem 2.12 for \( q(z) = \left( \frac{1 + z}{1 - z} \right)^\gamma, \) \( 0 < \gamma \leq 1. \)

Combining Theorem 2.9 and Theorem 2.12, we state the following sandwich theorem.

**Theorem 2.15** Let \( q_1 \) and \( q_2 \) be convex and univalent in \( U \) such that \( q_1(z) \neq 0 \) and \( q_2(z) \neq 0, \) for all
\( z \in U \). Suppose that \( q_1 \) satisfies (14) and \( q_2 \) satisfies (20). If \( f \in \mathcal{A} \), \( \left( \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)} \right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q \), \( \delta \in \mathbb{C}, \delta \neq 0 \) and \( \psi_{\lambda}^{m,n}(\alpha, \beta; z) \) is as defined in (15) univalent in \( U \), then
\[
\alpha q_1(z) + \beta \xi q_1(z) < \psi_{\lambda}^{m,n}(\alpha, \beta; z) \prec \alpha q_2(z) + \beta \xi q_2(z),
\]
for \( \alpha, \beta \in \mathbb{C}, \beta \neq 0 \), implies
\[
q_1(z) \prec \left( \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)} \right)^{\delta} \prec q_2(z), \ z \in U, \delta \in \mathbb{C}, \delta \neq 0,
\]
and \( q_1 \) and \( q_2 \) are respectively the best subordinate and the best dominant.

For \( q_1(z) = \frac{1 + A_1 z}{1 + B_1 z} \) and \( q_2(z) = \frac{1 + A_2 z}{1 + B_2 z} \), where \(-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1\), we have the following corollary.

**Corollary 2.16** Let \( m, n \in \mathbb{N} \), \( \lambda \geq 0 \). Assume that (14) and (20) hold for \( q_1(z) = \frac{1 + A_1 z}{1 + B_1 z} \) and
\[
q_2(z) = \frac{1 + A_2 z}{1 + B_2 z}, \text{ respectively. If } f \in \mathcal{A}, \left( \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)} \right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q \text{ and}
\]
\[
\alpha \frac{1 + A_2 z}{1 + B_2 z} + \beta \frac{(A_1 - B_1)^2 z}{(1 + B_1 z)^2} \prec \psi_{\lambda}^{m,n}(\alpha, \beta; z), \ z \in U,
\]
for \( \alpha, \beta \in \mathbb{C}, \beta \neq 0 \), \(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1\), where \( \psi_{\lambda}^{m,n} \) is defined in (7), then
\[
\frac{1 + A_1 z}{1 + B_1 z} \prec \left( \frac{SR^{m+1,n} f(z)}{SR^{m,n} f(z)} \right)^{\delta} \prec \frac{1 + A_2 z}{1 + B_2 z}, \ z \in U, \delta \in \mathbb{C}, \delta \neq 0,
\]
hence \( \frac{1 + A_1 z}{1 + B_1 z} \) and \( \frac{1 + A_2 z}{1 + B_2 z} \) are the best subordinate and the best dominant, respectively.

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