A NOTE ON STRONG LIMIT THEOREMS FOR PAIRWISE NQD RANDOM VARIABLES

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ABSTRACT
In this note, we point out that one of Sung’s results of strong law of large numbers for pairwise NQD random variables (Sung, Communications in Statistics-Theory and Methods, 2013, 42: 3965-3973) is a special case of one of Yang’s results of strong limit theorems for arbitrary stochastic sequences (Yang, J. Math. Anal. Appl, 2007, 326: 1445-1451).

Keywords: Strong limit theorems, arbitrary stochastic sequences, sequence of pairwise negative quadrant dependent (NQD) random variables.

1. INTRODUCTION
Let \( \{X_n, n \geq 1\} \) be a sequence of pairwise negative quadrant dependent (NQD) random variables (see[1]). Li and Yang [2] have obtained a strong convergence theorem for sequence of pairwise NQD random variables by using a Kolmogorov type generalized three series theorem. Sung [3] has proved a strong law of large numbers for pairwise NQD random variables by using the truncation technique and method of subsequences which generalized Li and Yang’s result. Yang [4] has obtained a strong limit theorem for arbitrary stochastic sequences. In this note, we point out that the Sung’s result is a special case the Yang’s result. So Yang’s result is a generalization of not only Sung’s result, but also Li and Yang’s result. We also give a simple proof of a generalization of Sung’s result. The following is our explanation.

Let \( \{X_n, \mathcal{F}_n, n \geq 0\} \) be a stochastic sequence on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), that is, the sequence of \( \sigma \)-fields \( \{\mathcal{F}_n, n \geq 0\} \) in \( \mathcal{F} \) is increasing in \( n \), and \( X_n \) is measurable \( \mathcal{F}_n \).

Lemma 1 (Yang, [4, Theorem 2]). Let \( \{X_n, \mathcal{F}_n, n \geq 0\} \) be a stochastic sequence defined as before, and \( \{c_n, n \geq 1\} \) be a sequence of non-zero random variables such that \( c_n \) is measurable \( \mathcal{F}_{n-1} \). Let \( \varphi_n : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be Borel functions and \( \alpha_n \geq 0, \beta_n \leq 1, K_n \geq 1, M_n \geq 1(n \in \mathbb{N}) \) be constants satisfying

\[
0 < x_1 \leq x_2 \implies \frac{\varphi_n(x_1)}{x_1^{\alpha_n}} \leq K_n \frac{\varphi_n(x_2)}{x_2^{\alpha_n}} \quad \text{and} \quad \frac{x_1^{\beta_n}}{\varphi_n(x_1)} \leq M_n \frac{x_2^{\beta_n}}{\varphi_n(x_2)}.
\]

Set

\[
A = \{ \omega : \sum_{n=1}^{\infty} K_n E[\varphi_n(|X_n|) | \mathcal{F}_{n-1}] / \varphi_n(|c_n|) < \infty \},
\]

\[
B = \{ \omega : \sum_{n=1}^{\infty} M_n E[\varphi_n(|X_n|) | \mathcal{F}_{n-1}] / \varphi_n(|c_n|) < \infty \}.
\]

Then
\[
\sum_{n=1}^{\infty} \frac{X_n}{c_n} \text{ converges a.e. on } A \cap B
\] (4)

2. MAIN RESULTS

**Theorem 1.** Let \( \{X_n, n \geq 1\} \) be arbitrary sequences of random variables, and \( \{a_n, n \geq 1\} \) be a sequence of positive constants with \( a_n \uparrow \infty \). Let \( \varphi_n : R_+ \to R_+ \) be non-decreasing functions and \( M_n \geq 1(n \geq 1) \) be constants satisfying

\[
0 < x_1 \leq x_2 \implies \frac{x_1}{\varphi_n(x_1)} \leq M_n \frac{x_2}{\varphi_n(x_2)}.
\] (5)

If

\[
\sum_{n=1}^{\infty} M_n E(\varphi_n(|X_n|)/\varphi_n(a_n)) < \infty,
\] (6)

then

\[
\sum_{n=1}^{\infty} \frac{X_n}{a_n} \text{ converges a.e.,}
\] (7)

and

\[
\lim_{n \to \infty} \frac{1}{a_n} \sum_{k=1}^{n} X_k = 0 \text{ a.e.}
\] (8)

**Proof.** By (6) and non-negativity of \( \varphi_n \), we have

\[
\sum_{n=1}^{\infty} M_n E(\varphi_n(|X_n|)/\varphi_n(a_n)) < \infty \text{ a.e.,}
\] (9)

and

\[
\sum_{n=1}^{\infty} E(\varphi_n(|X_n|)/\varphi_n(a_n)) < \infty \text{ a.e.,}
\] (10)

where \( \mathcal{F}_n = \sigma(X_1, \cdots, X_n) \) and \( F_0 = \{\emptyset, \Omega\} \). Letting \( \alpha_n = 0, \beta_n = 1 \) and \( K_n = 1 \) for all \( n \) in Lemma 1, we have \( P(A) = P(B) = 1 \), hence (7) holds, which together with the Kronecker lemma gives (8).

**Corollary 1** (Sung, [3, Theorem 1]). Let \( \{X_n, n \geq 1\} \) be a sequence of pairwise NQD random variables, and \( \{a_n, n \geq 1\} \) be a sequence of positive constants with \( a_n \uparrow \infty \). Let \( f_n : R_+ \to R_+ \) be non-decreasing functions and let \( p_n \geq 2 \) and \( H_n \geq 1 \) \( (n \in N) \) be constants satisfying

\[
0 < x_1 \leq x_2 \implies \frac{x_1^{p_n}}{f_n(x_1)} \leq H_n \frac{x_2^{p_n}}{f_n(x_2)}.
\] (11)

If

\[
\sum_{n=1}^{\infty} (H_n E f_n(|X_n|/f_n(a_n))^p_n < \infty,
\] (12)

then
Theorem 1. Let \( Y_n = X_n \mathbb{I}_{[X_n < a_n]} \). According to (5), we conclude
\[
\sum_{n=1}^{\infty} \mathbb{E} \left[ \frac{|Y_n|}{a_n} \right] = \sum_{n=1}^{\infty} \mathbb{E} \left[ \frac{|X_n|}{a_n} \mathbb{I}_{[X_n < a_n]} \right] \\
\leq \sum_{n=1}^{\infty} M_n \mathbb{E} \left[ \frac{\phi_n(|X_n|)}{\phi_n(a_n)} \right] I_{[X_n < a_n]} \]
\[
\leq \sum_{n=1}^{\infty} M_n \mathbb{E} \left[ \frac{\phi_n(|X_n|)}{\phi_n(a_n)} \right] < \infty,
\]
therefore
\[
\sum_{n=1}^{\infty} \frac{Y_n}{a_n} \text{ converges a.e.} \quad \tag{19}
\]
Since \( \varphi_n : R_+ \to R_+ \) are non-decreasing functions, thus

\[
\sum_{n=1}^{\infty} P(Y_n \neq X_n) = \sum_{n=1}^{\infty} P(|X_n| \geq a_n)
\]

\[
\leq \sum_{n=1}^{\infty} E\left[ \frac{\varphi_n(|X_n|)}{\varphi_n(a_n)} \mathbb{I}_{|X_n| \geq a_n} \right]
\]

\[
\leq \sum_{n=1}^{\infty} M_n E\left[ \frac{\varphi_n(|X_n|)}{\varphi_n(a_n)} \right] < \infty \tag{20}
\]

By (20) and the Borel-Cantelli lemma, we obtain

\[
\sum_{n=1}^{\infty} \frac{X_n - Y_n}{a_n} \text{ converges a.e..} \tag{21}
\]

(8) follows from (19) and (21), and (9) follows from (8) and the Kronecker lemma.

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REFERENCES


