Method for Solving Nonlinear Partial Differential Equations

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Abstract

In this paper, two-dimensional differential transform method is proposed to solve nonlinear Gas Dynamic and Klein-Gordon equations. The approximate solution of this equation is calculated in the form of a series with easily computable terms. The results reveal that the proposed method is very effective and simple.

Keywords: Two-dimensional differential transform; Gas Dynamic and Klein-Gordon equations.

1. Introduction

In this study, a new transformation called two-dimensional differential transform is introduced to solve nonlinear Gas Dynamic and Klein-Gordon equations. Jafari [5] used homotopy analysis method to solve the nonlinear Gas Dynamic equation. The concept of differential transform (one-dimension) was first proposed and applied to solve linear and nonlinear initial value problems in electric circuit analysis by Zhou [9]. Using one-dimensional differential transform, Chen and Ho [3] proposed a method to solve eigenvalue problems. Using two-dimensional differential transformation technique, a closed form series solution or an approximate solution can be obtained. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations. Ayaz developed differential transform method to two-dimensional problem for PDE’s initial value problems [1, 2]. Kurnaz et al. generalized DTM to n-dimensional case in order to solve PDEs [6]. Recently, this method has been successfully employed to solve many types of nonlinear problems in science and engineering [4, 7, 8]. This paper investigates for the first time the applicability and effectiveness of two-dimensional differential transform method on nonlinear Gas Dynamic and Klein-Gordon equations.

2. Two-Dimensional Differential Transform

The basic definitions and fundamental operations of the two-dimensional differential transform are defined in [4] as follows:

\[
W(k, h) = \frac{1}{k!h!} \left[ \frac{\partial^{k+h} w(x, y)}{\partial x^k \partial y^h} \right]_{(0,0)}, \tag{1}
\]

where \( w(x, y) \) is the original function and \( W(k, h) \) is the transformed function. The differential inverse transform of \( W(k, h) \) is defined as

\[
w(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k, h) x^k y^h \tag{2}
\]

and from Eqs. (1) and (2) can be concluded...
\[ w(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k!h!} \frac{\partial^{k+h} w(x, y)}{\partial x^k \partial y^h} |_{(0,0)} x^k y^h \]  

(3)

The following theorems that can be deduced from Eqs. (1) and (2) are given as:

**Theorem 1:** If \( w(x, y) = u(x, y) \pm v(x, y) \), then \( W(k, h) = U(k, h) \pm V(k, h) \).

**Theorem 2:** If \( w(x, y) = \alpha u(x, y) \) then \( W(k, h) = \alpha U(k, h) \).

**Theorem 3:** If \( w(x, y) = \frac{\partial u(x, y)}{\partial x} \) then \( W(k, h) = (k + 1)U(k + 1, h) \).

**Theorem 4:** If \( w(x, y) = \frac{\partial u(x, y)}{\partial y} \) then \( W(k, h) = (h + 1)U(k, h + 1) \).

**Theorem 5:** If \( w(x, y) = \frac{\partial^r u(x, y)}{\partial x^r \partial y^s} \) then \( W(k, h) = (k + 1)(k + 2)...(k + r)(h + 1)(h + 2)...(h + s)U(k + r, h + s) \).

**Theorem 6:** If \( w(x, y) = u(x, y)v(x, y) \), then \( W(k, h) = \sum_{r=0}^{k} \sum_{s=0}^{h} U(r, h - s)V(k - r, s) \).

**Theorem 7:** If \( w(x, y) = x^m y^n \) then \( W(k, h) = \delta(k - m, h - n) = \delta(k - m)\delta(h - n) \) where

\[
\delta(k - m) = \begin{cases} 
1, & \text{for } k = m, \quad h = n, \\
0, & \text{otherwise}
\end{cases}
\]

**Theorem 8:** If \( w(x, y) = \frac{\partial u(x, y)}{\partial x} \frac{\partial v(x, y)}{\partial x} \) then

\[
W(k, h) = \sum_{r=0}^{k} \sum_{s=0}^{h} (r + 1)(k - r + 1)U(r + 1, h - s)V(k - r + 1, s).
\]

**Theorem 9:** If \( w(x, y) = x^m \sin(at + b) \) then

\[ W(k, h) = \frac{a^b}{h!} \delta(k - m)\sin\left(\frac{h\pi}{2} + b\right). \]

**Theorem 10:** If \( w(x, y) = x^m \cos(at + b) \) then \( W(k, h) = \frac{a^b}{h!} \delta(k - m)\cos\left(\frac{h\pi}{2} + b\right) \).
Theorem 11: If \( w(x, y) = x^m e^{\alpha x} \) then \( W(k, h) = \frac{a^h}{h!} \delta(k - m) \)

The proofs of Theorems are available in Ref. [1].

3. APPLICATIONS

In this section we consider examples that demonstrate the performance and efficiency of the generalized differential transform method for solving nonlinear Gas Dynamic and Klein-Gordon equations.

Example 1 Consider the following homogeneous nonlinear Gas Dynamic equation

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} - u(1-u) = 0, \quad (4)
\]

with the initial condition

\[ u(x,0) = e^{-x} \quad (5) \]

whose exact solution can be expressed as

\[ u(x,t) = e^{-x} \quad (6) \]

Taking two-dimensional differential transform of Eq. (4), we obtain

\[
(h+1)U(k,h+1) = -\sum_{r=0}^{k} \sum_{s=0}^{h} (k-r+1)U(r,h-s)U(k-r+1,s) + U(k,h) - \sum_{r=0}^{k} \sum_{s=0}^{h} U(r,h-s)U(k-r,s). \quad (7)
\]

The related initial conditions should be also transformed as follows

\[ u(k,0) = \frac{(-1)^k}{k!} \quad k = 0,1,2,... \quad (8) \]

Substituting Eq. (8) into Eq. (7), and by recursive method, the result is listed as follows:

\[
\begin{align*}
U(0,1) & = 1 \\
U(1,1) & = -1 \\
U(0,2) & = \frac{1}{2!} \\
U(2,1) & = \frac{1}{2!} \\
U(1,2) & = -\frac{1}{2!}
\end{align*} \quad (9)
\]

Substituting all \( W(k, h) \) into Eq. (2), we have series solution as follows:

\[
u(x,t) = (1 - x + t - xt + \frac{t^2}{2!} + \frac{x^2}{2!} + \frac{x^2 t}{2!} - \frac{xt^2}{2!} + \cdots) = e^{-x} \quad (10)
\]
This is the exact solution.

**Example 2** Consider the following non-homogeneous nonlinear Gas Dynamic equation

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} - u(1 - u) = -e^{-x},
\]

with the initial condition,

\[
u(x,0) = 1 - e^{-x}.
\]

It’s exact solution can be expressed as

\[
u(x,t) = 1 - e^{-x}
\]

Taking two-dimensional differential transform of Eq. (11), we obtain

\[
(h + 1)U(k, h + 1) = -\sum_{r=0}^{k} \sum_{s=0}^{h} (k - r + 1)U(r, h - s)U(k - r + 1, s) + U(k, h) - \sum_{r=0}^{k} \sum_{s=0}^{h} U(r, h - s)U(k - r, s) - \frac{(-1)^k}{k!} \frac{1}{h!}.
\]

The related initial conditions should be also transformed as follows

\[
u(k,0) = \frac{(-1)^{k+1}}{k!} \\
u(0,0) = 0
\]

Substituting Eq. (15) into Eq. (14), and by recursive method, the result is listed as follows:

\[
U(0,1) = -1 \\
U(0,2) = \frac{-1}{2!} \\
U(1,2) = \frac{1}{2!} \\
U(0,3) = \frac{-1}{3!} \\
U(1,3) = \frac{1}{3!} \\
U(2,3) = \frac{-1}{3!2!}
\]

Substituting all \( W(k, h) \) into Eq. (2), we have series solution as follows:

\[
u(x,t) = (x - t + xt - \frac{t^2}{2!} - \frac{x^2}{2!} + \frac{x^2t}{2!} + \frac{x^3}{3!} - \frac{t^3}{3!} + \frac{xt^2}{3!2!} - \frac{x^3t}{3!2!} + \frac{x^2t^2}{3!2!} - \frac{x^2t^3}{3!2!} + \ldots) = 1 - e^{-x}
\]

which is the exact solution.

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Example 3: Consider the following non-homogeneous nonlinear Klein-Gordon equation

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\pi^2}{4} u + u^2 = x^2 \sin^2 \frac{\pi t}{2}
\]  

(17)

with the initial condition

\[
u(x,0) = 0, \quad u_t(x,0) = \frac{\pi x}{2}
\]  

(18)

whose exact solution can be expressed as

\[
 u(x,t) = x \sin \frac{\pi t}{2}.
\]  

(19)

Taking two-dimensional differential transform of Eq. (17), we obtain

\[
(h + 1)(h + 2)U(k, h + 2) = (k + 1)(k + 2)U(k + 2, h) - \frac{\pi^2}{4} U(k, h) - \sum_{r=0}^{k} \sum_{s=0}^{h} U(r, h - s)U(k - r, s)
\]

\[
+ \sum_{r=0}^{k} \sum_{s=0}^{h} \delta(r - 2)\delta(s - 0)\sum_{p=0}^{h-r} \sum_{s=0}^{r} \delta(t - 0)\delta((k - r - t) - 0) \frac{(\frac{\pi}{2})^p (\frac{\pi}{2})^{(h-r-p)}}{p!(h-s-p)!}\sin \frac{\pi p}{2}\sin \frac{(h-s-p)\pi}{2}
\]  

(20)

The related initial conditions should be also transformed as follows

\[
u(k,0) = 0, \quad k = 0, 2, 3, \ldots, \quad u(1,1) = \frac{\pi}{2}
\]  

(21)

Substituting Eq. (21) into Eq. (20), and by recursive method, the result are listed as follows:

\[
\begin{align*}
U(0,1) &= 0 & U(0,2) &= 0 \\
U(2,1) &= 0 & U(1,2) &= 0 \\
U(0,3) &= 0 & U(2,2) &= 0 \\
U(1,3) &= \frac{\pi^3}{3!2^3} & U(3,1) &= 0 \\
U(2,3) &= 0 & U(3,2) &= 0 \\
U(1,4) &= 0 & U(4,1) &= 0 \\
U(2,4) &= 0 & U(4,2) &= 0 \\
U(3,4) &= 0 & U(4,3) &= 0 \\
U(4,4) &= 0 & U(1,5) &= \frac{\pi^5}{5!2^5}
\end{align*}
\]  

(22)

Substituting all W(k,h) into Eq. (2), we have series solution as follows:
\[ u(x,t) = x \left( \frac{\pi}{2} - \frac{(\pi)^3}{3!2^3} + \frac{(\pi)^5}{5!2^5} + \cdots \right) = x \sin \frac{\pi t}{2} \]

which is the exact solution.

4. CONCLUSIONS

Two-dimensional differential transform have been applied to obtain the solution of non-linear Gas Dynamic and Klein-Gordon equations. The present method reduces the computational difficulties of the other methods and all the calculations can be made simply. On the other hand the results are quite reliable. The present study has confirmed that the differential transform method offers great advantages of straightforward applicability, computational efficiency and high accuracy.

5. REFERENCES


