THE ORLICZ SPACE OF $\chi$

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ABSTRACT

In this paper we introduced Orlicz space of $\chi$. We establish some inclusion relations, topological results and we characterize the duals of these Orlicz of $\chi$ sequence spaces.

2000 AMS Subject Classification: 40A05 , 40C05 , 40D05. 
Keywords: $\chi$-sequence , analytic sequence , Orlicz function.

INTRODUCTION

A complex sequence, whose $k^{th}$ term is $x_k$ is denoted by $\{x_k\}$ or simply $x$. Let $\Phi$ be the set of all finite sequences. A sequence $x = \{x_k\}$ is said to be analytic if $\sup_{(k)} |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by $\chi$. $\chi$ was discussed in Kamthan [21]. Matrix transformation involving $\chi$ were characterized by Sridhar [22] and Sirajiudeen [23]. Let $\chi$ be the set of all those sequences $x = (x_k)$ such that $\left(\angle k|x_k|)^{1/k} \rightarrow 0$ as $k \rightarrow \infty$. Then $\chi$ is a metric space with the metric

$$d(x, y) = \sup_{(k)} \left(\angle k|x_k| - y_k|^{1/k} : k = 1, 2, 3, \ldots \right)$$

Orlicz [1] used the idea of Orlicz function to construct the space ($L^M$). Lindenstrauss and Tzafriri [2] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space $\ell^M$ contains a subspace isomorphic to $\ell_p$ ($1 \leq p < \infty$). Subsequently different classes of sequence spaces defined by Parashar and Choudhary [3], Mursaleen et al [4], Bektas and Altin [5], Tripathy et al. [6]. Rao and Subramanian [7] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref [8].

Recall (11],[8]) an Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. If convexity of Orlicz function $M$ is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function, defined and discussed by Ruckle [9] and Maddox [10]. Let $(\Omega, \Sigma, \mu)$ be a finite measure space. We denote by $E(\mu)$ the space of all (equivalence classes of) $\Sigma$-measurable functions $x$ from $\Omega$ into $[0, \infty)$. Given an Orlicz function $M$, we define on $E(\mu)$ a convex functional $I_M$ by

$$I_M (x) = \int_{\Omega} M(x(t))d\mu,$$

and an Orlicz space $L^M(\mu)$ by $L^M(\mu) = \{ x \in E(\mu) : I_M (\lambda x) < +\infty \ \text{for some} \ \lambda > 0 \}$. (For detail see [1], [8]).

Lindenstrauss and Tzafriri [2] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell^M = \left\{ x \in w : \sum_{k = 1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty, \ \text{for some} \ \rho > 0 \right\}.$$
where \( w = \{ \text{all complex sequences} \} \).

The space \( \ell_M \) with the norm

\[
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\},
\]

becomes a Banach space which is called an Orlicz space. For \( M(t) = t^p, 1 \leq p < \infty \), the spaces \( \ell_M \)
coincide with the classical sequence space \( \ell_p \).

Given a sequence \( x = \{x_k\} \) its \( n \)th section is the sequence \( x^{(n)} = \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\} \).

\( s^{(k)} = \left( 0,0,\ldots, \frac{1}{\sqrt[k]{k}},0,0,\ldots \right) \frac{1}{\sqrt[k]{k}} \) in the \( n \)th place and zero’s else where; An FK-space (Frechet coordinate
space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate
functionals \( p_k(x) = x_k \) \((k = 1,2,\ldots)\) are continuous.

We recall the following definitions [see [19] ] An FK-space is a locally convex Frechet space which is made up of
sequences and has the property that coordinate projections are continuous. A metric space \( (X,d) \) is said to have
AK (or Sectional convergence) if and only if \( d(x^{(n)},x) \to 0 \) as \( n \to \infty \) [see [19] ]. The space is said to have AD
(or) be an AD space if \( \Phi \) is dense in \( X \). We note that AK implies AD by [18].

If \( X \) is a sequence space, we define

(i) \( X^\odot \) = the continuous dual of \( X \).
(ii) \( X^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for each } x \in X\} \);
(iii) \( X^\beta = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \) is convergent, for each \( x \in X\} \);
(iv) \( X^\gamma = \{a = (a_k) : \sup_{\{n\}} \left| \sum_{k=1}^{n} a_k x_k \right| < \infty, \text{ for each } x \in X\} \).
(v) Let \( X \) be an FK-space \( \supset \Phi \). Then \( X^f = \{f(x^{(n)}) : f \in X^\beta \} \).

\( X^\alpha, X^\beta, X^\gamma \) are called the
\( \alpha \) - (or \( K\beta \) the-Toeplitz), dual of \( X \).
\( \beta \) - (or generalized \( K\beta \) the-Toeplitz), dual of \( X \).
\( \gamma \) - dual of \( X \).

Note that \( X^\alpha \subset X^\beta \subset X^\gamma \).

If \( X \subset Y \) then \( Y^\mu \subset X^\mu \), for \( \mu = \alpha, \beta, \text{ or } \gamma \).

**Lemma 1:** (See [19,Theorem 7.2.7])

Let \( X \) be an FK-space \( \supset \Phi \).

Then

(i) \( X^\gamma \subset X^f \).
(ii) If \( X \) has AK, \( X^\beta = X^f \).
(iii) If \( X \) has AD, \( X^\beta = X^\gamma \).

**Definition 1.1:** The space consisting of all those sequences \( x \) in \( w \) such that

\[
M \left( \frac{\sum_{k=1}^{\infty} |x_k|^{\gamma_k}}{\rho} \right) \to 0 \text{ as } k \to \infty
\]

for some arbitrary fixed \( \rho \) > 0 is denoted by \( X_M \).
\( M \) being an Orlicz function. \( \mathcal{X}_M \) is called the Orlicz space of \( \mathcal{X} \).

The space \( \mathcal{X}_M \) is a metric space with the metric 
\[
d(x, y) = \sup_{(k)} \left( \frac{\langle k \rangle |x_k - y_k|^\gamma}{\rho} \right)
\]
for all \( x = \{x_k\} \) and \( y = \{y_k\} \) in \( \mathcal{X}_M \).

**Definition 1.2.** The space consisting of all those sequences \( x \) in \( w \) such that 
\[
\sup_k M \left( \frac{|x_k|^\gamma}{\rho} \right) < \infty
\]
for some arbitrarily fixed \( \rho > 0 \) is denoted by \( \mathcal{M}_M \), \( M \) being a modulus function. In other words, \( \left\{ M \left( \frac{|x_k|^\gamma}{\rho} \right) \right\} \) is a bounded sequence. \( \mathcal{M}_M \) is called the Orlicz space of bounded sequences.

2. **MAIN RESULTS:**

**PROPOSITION 2.3:** \( \mathcal{X}_M \subset \Gamma_M \), with the hypothesis that \( M \left( \frac{|x_k|^\gamma}{\rho} \right) \leq M \left( \frac{\langle k \rangle |x_k|^\gamma}{\rho} \right) \)

**PROOF:**

Let \( x \in \mathcal{X}_M \). Then we have the following implications

(2.2) \[
M \left( \frac{\langle k \rangle |x_k|^\gamma}{\rho} \right) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.
\]

But \( M \left( \frac{|x_k|^\gamma}{\rho} \right) \leq M \left( \frac{\langle k \rangle |x_k|^\gamma}{\rho} \right) \), by our assumption, implies that

\[
\Rightarrow M \left( \frac{|x_k|^\gamma}{\rho} \right) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty, \quad \text{by} \ (2.2).
\]

(2.3) \[
\Rightarrow x \in \Gamma_M
\]

\[
\Rightarrow \mathcal{X}_M \subset \Gamma_M.
\]

This completes the proof.
PROPOSITION 2.4: \( \chi_M \) has AK where \( M \) is a modulus function.

PROOF:

Let \( x = \{x_k\} \in \chi_M \), but then \( \left\{ M \left( \frac{\langle k \rangle |x_k|^{\gamma_k}}{\rho} \right) \right\} \in \chi \), and hence

\[
\sup_{k \geq n+1} M \left( \frac{\langle k \rangle |x_k|^{\gamma_k}}{\rho} \right) \to 0 \quad \text{as} \quad n \to \infty \quad \text{(2.4)}
\]

\[
d(x, x^{[n]}) = \sup_{k \geq n+1} M \left( \frac{\langle k \rangle |x_k|^{\gamma_k}}{\rho} \right) \to 0 \quad \text{as} \quad n \to \infty,
\]

by using (2.4).

\[
\Rightarrow x^{[n]} \to x \quad \text{as} \quad n \to \infty,
\]

implying that \( \chi_M \) has AK.

This completes the proof.

PROPOSITION 2.5: \( \chi_M \) is solid

PROOF:

Let \( |x_k| \leq |y_k| \) and let \( y = (y_k) \in \chi_M \).

\[
M \left( \frac{\langle k \rangle |x_k|^{\gamma_k}}{\rho} \right) \leq M \left( \frac{\langle k \rangle |y_k|^{\gamma_k}}{\rho} \right), \quad \text{because} \quad M \quad \text{is} \quad \text{non-decreasing}.
\]

But \( M \left( \frac{\langle k \rangle |y_k|^{\gamma_k}}{\rho} \right) \in \chi \), because \( y \in \chi_M \).

That is \( M \left( \frac{\langle k \rangle |y_k|^{\gamma_k}}{\rho} \right) \to 0 \quad \text{as} \quad k \to \infty \).

and \( M \left( \frac{\langle k \rangle |x_k|^{\gamma_k}}{\rho} \right) \to 0 \quad \text{as} \quad k \to \infty \).
Therefore \( x = \{ x_k \} \in \mathcal{X}_M \).

This completes the proof.

**PROPOSITION 2.6:** Let \( M \) be an Orlicz function which satisfies \( \Delta_2 \)-condition.

Then \( \mathcal{X} \subset \mathcal{X}_M \).

**PROOF:**

(2.5) Let \( x \in \mathcal{X} \)

Then \( \left( \angle k |x_k| \right)^{\frac{1}{k}} \leq \varepsilon \) sufficiently large \( k \) and every \( \varepsilon > 0 \). But then by taking \( \rho \geq \frac{1}{2} \)

\[
M \left( \frac{\left( \angle k |x_k| \right)^{\frac{1}{k}}}{\rho} \right) \leq M \left( \frac{\varepsilon}{\rho} \right) \quad \text{(because } M \text{ is non-decreasing)}
\]

\[
\leq M(2\varepsilon)
\]

\[\Rightarrow \quad M \left( \frac{\left( \angle k |x_k| \right)^{\frac{1}{k}}}{\rho} \right) \leq K M(\varepsilon) \quad \text{by } \Delta_2 \text{-condition, for some } K > 0. \quad (2.6)\]

\[
\leq \varepsilon
\]

\[\Rightarrow \quad M \left( \frac{\left( \angle k |x_k| \right)^{\frac{1}{k}}}{\rho} \right) \rightarrow 0 \text{ as } k \rightarrow \infty \quad \text{(by defining } M(\varepsilon) < \frac{\varepsilon}{k} \text{)}
\]

Hence \( x \in \mathcal{X}_M \).

From (2.5) and since

(2.7) \( x \in \mathcal{X}_M \).

we get

(2.8) \( \mathcal{X} \subset \mathcal{X}_M \)

This completes the proof.

**PROPOSITION 2.7:** If \( M \) is a modulus function, then

\( \mathcal{X}_M \) is linear set over the set of complex numbers \( c \).
PROOF:

Let \( x, y \in \chi_M \) and \( \alpha, \beta \in c \)

In order to prove the result we need to find some \( \rho_3 \) such that

\[
M\left( \frac{(\angle k |\alpha x_k + \beta y_k|)^{\frac{1}{k}}}{\rho_3} \right) \to 0 \quad \text{as} \quad k \to \infty.
\]

Since \( x, y \in \chi_M \), there exists some positive \( \rho_1 \) and \( \rho_2 \) such that

\[
M\left( \frac{(\angle k |x_k|)^{\frac{1}{k}}}{\rho_1} \right) \to 0 \quad \text{as} \quad k \to \infty \quad \text{and}
\]

\[
M\left( \frac{(\angle k |y_k|)^{\frac{1}{k}}}{\rho_2} \right) \to 0 \quad \text{as} \quad k \to \infty .
\]

Since \( M \) is a non-decreasing modulus function, we have

\[
M\left( \frac{(\angle k |\alpha x_k + \beta y_k|)^{\frac{1}{k}}}{\rho_3} \right) \leq M\left( \frac{(\angle k |\alpha x_k|)^{\frac{1}{k}}}{\rho_3} + \frac{(\angle k |\beta y_k|)^{\frac{1}{k}}}{\rho_3} \right)
\]

\[
\leq M\left( \frac{(\angle k |x_k|)^{\frac{1}{k}}}{\rho_3} + \frac{(\angle k |y_k|)^{\frac{1}{k}}}{\rho_3} \left| \alpha \right| \frac{(\angle k |x_k|)^{\frac{1}{k}}}{\rho_3} + \frac{(\angle k |y_k|)^{\frac{1}{k}}}{\rho_3} \right)
\]

Take \( \rho_3 \) such that

\[
\frac{1}{\rho_3} = \min \left\{ \frac{1}{|\alpha|}, \frac{1}{\rho_1}, \frac{1}{|\beta|}, \frac{1}{\rho_2} \right\}
\]

Then

\[
M\left( \frac{(\angle k |\alpha x_k + \beta y_k|)^{\frac{1}{k}}}{\rho_3} \right) \leq M\left( \frac{(\angle k |x_k|)^{\frac{1}{k}}}{\rho_1} + \frac{(\angle k |y_k|)^{\frac{1}{k}}}{\rho_2} \right)
\]
\[ (2.13) \quad \leq M \left( \frac{\angle k |x_k|}{\rho_1} \right)^{\frac{1}{\rho_1}} + M \left( \frac{\angle k |y_k|}{\rho_2} \right)^{\frac{1}{\rho_2}} \]

\[ \rightarrow 0 \quad \text{(by (2.10))}. \]

\[ (2.14) \quad \text{Hence} \quad M \left( \frac{\angle k |\alpha x_k + \beta y_k|}{\rho_3} \right)^{\frac{1}{\rho_3}} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \]

So \((\alpha x + \beta y) \in \chi_M\). Therefore \(\chi_M\) is linear.

This completes the proof.

**DEFINITION 2.8:**

Let \( p = (p_k) \) be any sequence of positive real numbers. Then we define

\[ \chi_m(p) = \left\{ x = \{x_k\} : M \left( \frac{\angle k |x_k|}{\rho} \right)^{\frac{1}{\rho}} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \right\}. \quad (2.15) \]

Suppose that \( p_k \) is a constant for all \( k \), then \( \chi_m(p) = \chi_m \).

**PROPOSITION 2.9:** Let \( 0 \leq p_k \leq q_k \) and let \( \left\{ \frac{q_k}{p_k} \right\} \) be bounded. Then \( \chi_m(q) \subset \chi_m(p) \).

**PROOF:**

\[ (2.16) \quad \text{Let} \quad x \in \chi_m(q) \]

\[ (2.17) \quad \left( M \left( \frac{\angle k |x_k|}{\rho} \right)^{\frac{1}{\rho}} \right) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \]

Let \( t_k = \left( M \left( \frac{\angle |x_k|}{\rho} \right)^{\frac{1}{\rho}} \right)^{\frac{1}{q_k}} \) and \( \lambda_k = \frac{p_k}{q_k} \)

Since \( p_k \leq q_k \), we have \( 0 \leq \lambda_k \leq 1 \)

Take \( 0 < \lambda < \lambda_k \)
Define \( u_k = t_k \) \( (t_k \geq 1) \)

\[
= 0 \quad (t_k < 1)
\]

(2.18)

and \( v_k = 0 \) \( (t_k \geq 1) \)

\[
= t_k \quad (t_k < 1)
\]

\[
t_k = u_k + v_k
\]

\[
t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}
\]

Now it follows that

(2.19)

\[
u_k^{\lambda_k} \leq u_k \leq t_k \quad \text{and} \quad v_k^{\lambda_k} \leq v_k
\]

Since \( t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k} \), then \( t_k^{\lambda_k} \leq t_k + v_k^{\lambda_k} \)

\[
\left(M \left( \frac{\langle k | x_k \rangle^{\lambda_k}}{\rho} \right) \right)^{\lambda_k} \leq \left(M \left( \frac{\langle k | x_k \rangle^{\lambda_k}}{\rho} \right) \right)^{\lambda_k}
\]

(2.20)

\[
\Rightarrow \left(M \left( \frac{\langle k | x_k \rangle^{\lambda_k}}{\rho} \right) \right)^{\lambda_k} \leq \left(M \left( \frac{\langle k | x_k \rangle^{\lambda_k}}{\rho} \right) \right)^{\lambda_k}
\]

\[
\Rightarrow \left(M \left( \frac{\langle k | x_k \rangle^{\lambda_k}}{\rho} \right) \right)^{\lambda_k} \leq \left(M \left( \frac{\langle k | x_k \rangle^{\lambda_k}}{\rho} \right) \right)^{\lambda_k}
\]

But \( \left(M \left( \frac{\langle k | x_k \rangle^{\lambda_k}}{\rho} \right) \right)^{\lambda_k} \rightarrow 0 \) (by (2.17))

Hence \( \left(M \left( \frac{\langle k | x_k \rangle^{\lambda_k}}{\rho} \right) \right)^{\lambda_k} \rightarrow 0 \) as \( k \rightarrow \infty \)

(2.21)

Hence \( x \in \mathcal{X}_M (p) \)

From (2.16) and (2.21) we get

(2.22) \( \mathcal{X}_M (q) \subset \mathcal{X}_M (p) \)

This completes the proof.
PROPOSITION 2.10: Let $0 < \inf p_k \leq p_k \leq 1$. Then $\chi_M (p) \subset \chi_M$

(b) Let $1 \leq p_k \leq \sup p_k < \infty$. Then $\chi_M \subset \chi_M (p)$

PROOF (a):

Let $x \in \chi_M (p)$

(2.23) \[ M \left( \frac{\langle k \rangle |x_k|^{\frac{1}{p_k}}} {\rho} \right)^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty \]

Since $0 < \inf p_k \leq p_k \leq 1$

(2.24) \[ \left( M \left( \frac{\langle k \rangle |x_k|^{\frac{1}{p_k}}} {\rho} \right)^{p_k} \right) \leq \left( M \left( \frac{\langle k \rangle |x_k|^{\frac{1}{p}}} {\rho} \right)^{p_k} \right) \]

From (2.23) and (2.24) it follows that

(2.25) \[ x \in \chi_M \]

Thus

(2.26) \[ \chi_M (p) \subset \chi_M \]

We have thus proven (a).

(b)

Let $p_k \geq 1$ for each $k$ and $\sup p_k < \infty$

Let $x \in \chi_M$

(2.27) \[ M \left( \frac{\langle k \rangle |x_k|^{\frac{1}{p_k}}} {\rho} \right) \rightarrow 0 \text{ as } k \rightarrow \infty \]

Since $1 \leq p_k \leq \sup p_k < \infty$ we have

(2.28) \[ \left( M \left( \frac{\langle k \rangle |x_k|^{\frac{1}{p_k}}} {\rho} \right)^{p_k} \right) \leq \left( M \left( \frac{\langle k \rangle |x_k|^{\frac{1}{p_k}}} {\rho} \right)^{p_k} \right) \]
\[ \left( M \left( \frac{\langle k \mid x_k \rangle^{\frac{1}{k}}}{\rho} \right) \right)^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty \text{ (by using (2.27)).} \]

Therefore \( x \in \chi_M(p) \)

This completes the proof.

**PROPOSITION 2.11:** Let \( 0 < p_k \leq q_k < \infty \) for each \( k \). Then \( \chi_M(p) \subseteq \chi_M(q) \).

**PROOF:**

Let \( x \in \chi_M(p) \)

\[ (2.29) \quad \left( M \left( \frac{\langle k \mid x_k \rangle^{\frac{1}{k}}}{\rho} \right) \right)^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty \]

This implies that \( \left( M \left( \frac{\langle k \mid x_k \rangle^{\frac{1}{k}}}{\rho} \right) \right) \leq 1 \) for sufficiently large \( k \).

Since \( M \) is non-decreasing,

we get

\[ (2.30) \quad \left( M \left( \frac{\langle k \mid x_k \rangle^{\frac{1}{k}}}{\rho} \right) \right)^{q_k} \leq \left( M \left( \frac{\langle k \mid x_k \rangle^{\frac{1}{k}}}{\rho} \right) \right)^{p_k} \]

\[ \Rightarrow \left( M \left( \frac{\langle k \mid x_k \rangle^{\frac{1}{k}}}{\rho} \right) \right)^{q_k} \rightarrow 0 \text{ as } k \rightarrow \infty \text{ (by using (2.29)).} \]

\( x \in \chi_M(q) \)

Hence \( \chi_M(p) \subseteq \chi_M(q) \)

This completes the proof.

**PROPOSITION 2.12:** \( \chi_M(p) \) is \( r \)-convex for all \( r \) where \( 0 \leq r \leq \inf p_k \).

Moreover if \( p_k = p \leq 1 \forall k \), then they are \( p \)-convex.
PROOF:

We shall prove the Theorem for $\chi_M(p)$.

Let $x \in \chi_M(p)$ and $r \in \left(0, \liminf_{n \to \infty} p_n\right]$.

Then, there exists $k_0$ such that $r \leq p_k \forall k > k_0$.

Now, define

$$g^*(x) = \inf \left\{ \rho : M \left( \frac{\left\langle k|x_k - y_k|^{1/k} \right\rangle}{\rho} \right)^r + M \left( \frac{\left\langle k|x_k - y_k|^{1/k} \right\rangle}{\rho} \right)^{\mu} \right\}$$

(2.31)

Since $r \leq p_k \leq 1 \forall k > k_0$,

$g^*$ is subadditive :

Further, for $0 \leq |\lambda| \leq 1$

(2.32) \quad $|\lambda|^p \leq |\lambda|^r \forall k > k_0$.

Therefore, for each $\lambda$ we have

(2.33) \quad $g^*(\lambda x) \leq |\lambda|^r \cdot g^*(x)$

Now, for $0 < \delta < 1$,

(2.34) \quad $U = \left\{ x : g^*(x) \leq \delta \right\}$ which is an absolutely $r$-convex set, for

(2.35) \quad $|\lambda|^r + |\mu|^r \leq 1$ and $x, y \in U$.

Now

$$g^*(\lambda x + \mu y) \leq g^*(\lambda x) + g^*(\mu y)$$

$$\leq |\lambda|^r \cdot g^*(x) + |\mu|^r \cdot g^*(y)$$

$$\leq |\lambda|^r \delta + |\mu|^r \delta \quad \text{(using (2.33) and (2.34))}$$

(2.36) \quad $\leq \left(|\lambda|^r + |\mu|^r\right) \delta$

$$\leq 1 \cdot \delta \quad \text{by using (2.35).}$$

$$\leq \delta.$$
If \( p_k = p \leq 1 \ \forall k \) then for \( 0 < r < 1 \),

\[
U = \{ x : g^* (x) \leq \delta \}
\]
is an absolutely \( p \)-convex set.

This can be obtained by a similar analysis and therefore we omit the details.

This completes the proof.

**PROPOSITION 2.13:** \((\chi_M, ) = \wedge\)

**PROOF:**

Step 1.

\( \chi_M \subset \Gamma_M \) by proposition 2.3;

\[
\Rightarrow (\Gamma_M, ) \subset (\chi_M, ) \wedge \quad \text{But } (\Gamma_M, ) = \wedge \quad \text{[See [14]]}
\]

\( (\chi_M, ) \wedge \supset (\chi_M, ) \)

Step 2.

Let \( y \in (\chi_M, ) \)

\[
f(x) = \sum_{k=1}^{\infty} x_k y_k \quad \text{with } x \in \chi_M.
\]

We recall that \( s^{(k)} \) has \( \frac{1}{\angle k} \) in the \( k^{th} \) place and zero’s elsewhere, with

\[
x = s^{(k)}, \left\{ M \left( \frac{\angle k |x_k|}{\rho} \right) \right\} = \left\{ M \left( \frac{0^1}{\rho} \right), M \left( \frac{0^{1/2}}{\rho} \right), \ldots, M \left( \frac{1^{1/k}}{\rho} \right), M \left( \frac{0^{1/(k+1)}}{\rho} \right), \ldots \right\}
\]

\[
= \{ 0, 0, \ldots, M \left( \frac{1^{1/k}}{\rho} \right), 0, \ldots \}
\]

which converges to zero. Hence \( s^{(k)} \in \chi_M \).

Hence \( d(s^{(k)}, 0) = 1 \). There fore

But \( |y_k| \leq \| f \| d(s^{(k)}, 0) < \infty \ \forall k \).

Thus \( (y_k) \) is a bounded sequence and hence an analytic sequence. In other words \( y \in \wedge \).
(2.38) \[(X_M)^\mu \subset \wedge\]

Step 3.

From (2.37) and (2.38) we obtain

(2.39) \[(X_M)^\beta = \wedge.\]

This completes the proof.

**PROPOSITION 2.14:** \[(X_M)^\mu = \wedge \quad \text{for } \mu = \alpha, \beta, \gamma, f.\]

(Step 1)

\[X_M\] has AK by proposition 2.4. Hence by Lemma 2.2 (i)

we get \[(X_M)^\beta = (X_M)^\beta.\]

But \[(X_M)^\beta = \wedge.\]

(2.40) Hence \[(X_M)^\beta = \wedge.\]

(Step 2)

Since AK \Rightarrow AD. Hence by Lemma 2.2 (i i i)

we get \[(X_M)^\beta = (X_M)^\beta.\]

(2.41) Therefore \[(X_M)^\beta = \wedge.\]

(Step 3)

\[X_M\] is normal by proposition 5. Hence by Proposition 2.7 [2].

(2.42) we get \[(X_M)^\mu = (X_M)^\mu = \wedge.\]

From (2.40), and (2.42) we have

(2.43) \[(X_M)^\mu = (X_M)^\beta = (X_M)^\gamma = (X_M)^\gamma = \wedge.\]

**PROPOSITION 2.15:** The dual space of \[X_M\] is \(\wedge\). In other words \(X_M^* = \wedge\).

**PROOF:**

We recall that \(s^{(k)}\) has \(\frac{1}{\angle k}\) in the \(k^{th}\) place zeros else where , with
\[ x = s^k, \left\{ M \left( \frac{\angle k \cdot x_1 | Y_k}{\rho} \right) \right\} = \left\{ M(0)^{Y_1}, \frac{M(0)^{Y_2}}{\rho}, \ldots, \frac{M(1)^{Y_k}}{\rho}, \frac{M(0)^{Y_{k+1}}}{\rho}, \ldots \right\} \]

\[ = \left\{ 0, 0, \ldots, \frac{M(1)^{Y_k}}{\rho}, 0, \ldots \right\} \]

Hence \( s^{(k)} \in \chi_M \).

\[ f(x) = \sum_{k=1}^{\infty} x_k y_k \text{ with } x \in \chi_M \text{ and } f \in \chi_M^*, \text{ where } \chi_M^* \text{ is the dual space of } \chi_M. \]

Take \( x = s^{(k)} \in \chi_M. \) Then

\[
(2.44) \quad |y_k| \leq \|f\| d\left(s^{(k)}, 0\right) < \infty \text{ for all } k.
\]

Thus \( (y_k) \) is a bounded sequence and hence an analytic sequence.

In other words, \( y \in \wedge. \) Therefore \( \chi_M^* = \wedge. \)

This completes the proof.

**Lemma 2.16:** [19, Theorem 8.6.1] \( Y \supset X \iff Y^f \subset X^f \) where \( X \) is an AD-space and \( Y \) an FK-space.

**Proposition 2.17.** Let \( Y \) be any FK-space \( \supset \Phi. \) Then \( Y \supset \chi_M \) if and only if

the sequence \( s^{(k)} \) is weakly analytic.

**Proof:**

The following implications establish the result.

\( Y \supset \chi_M \iff Y^f \subset (\chi_M)^f, \text{ since } \chi_M \text{ has AD and by Lemma 2.16}. \)

\( \iff Y^f \subset \wedge, \text{ since } (\chi_M)^f = \wedge. \)

\[
(2.45) \quad \iff \text{for each } f \in Y^*, \text{ the topological dual of } Y. f\left(s^{(k)}\right) \in \wedge.
\]

\( \iff f\left(s^{(k)}\right) \text{ is analytic.} \)

\( \iff s^{(k)} \text{ is weakly analytic.} \)

This completes the proof.
PROPOSITION 2.18: \( \chi_M \) is a complete metric space under the metric

\[
d(x, y) = \sup_{(k)} \left\{ M \left( \frac{\angle k|x_k - y_k|^{\frac{1}{k}}}{\rho} \right) : k = 1, 2, 3, \ldots \right\}
\]

where \( x = (x_k) \in \chi_M \) and \( y = (y_k) \in \chi_M \).

PROOF:

Let \( \{x^{(n)}\} \) be a cauchy sequence in \( \chi_M \).

Then given any \( \varepsilon > 0 \) there exists a positive integer \( N \) depending on \( \varepsilon \) such that \( d\left(x^{(n)}, x^{(m)}\right) < \varepsilon \), for all \( n \geq N \) and for all \( m \geq N \).

Hence

\[
\sup_{(k)} \left\{ M \left( \frac{\angle k|x_k^{(n)} - x_k^{(m)}|^{\frac{1}{k}}}{\rho} \right) \right\} < \varepsilon \quad \text{for all } n \geq N \text{ and for all } m \geq N.
\]

Consequently \( \left\{ M \left( \frac{\angle k|x_k^{(n)}|}{\rho} \right) \right\} \) is a cauchy sequence in the metric space \( C \) of complex numbers. But \( C \) is complete. So,

\[
M \left( \frac{\angle k|x_k^{(n)}|}{\rho} \right) \to M \left( \frac{\angle k|x_k|}{\rho} \right), \quad \text{as } n \to \infty.
\]

Hence there exists a positive integer \( n_0 \) such that

\[
\sup_{(k)} \left\{ M \left( \frac{\angle k|x_k^{(n)} - x_k^{(m)}|^{\frac{1}{k}}}{\rho} \right) \right\} < \varepsilon \quad \text{for all } n \geq n_0.
\]

In particular, we have
Now
\[
M \left( \frac{\langle k | x_k^{(n)} - x_k \rangle}{\rho} \right) < \varepsilon
\]

Thus
\[
M \left( \frac{\langle k | x_k \rangle}{\rho} \right) < \varepsilon \quad \text{as } k \to \infty.
\]

That is
\[
(x_k) \in \chi_M.
\]

Therefore, \(\chi_M\) is a complete metric space.

This completes the proof.

REFERENCES


[23] S.M. Sirajudeen, Matrix transformations of \( c_0(p), \ell_\infty(p), \ell^p \) and \( \ell \) into \( \chi \), Indian J. Pure Math. 12(9) (1981): p.1106-1113.
