DYNAMICS OF A TEST PARTICLE IN THE RESTRICTED FOUR BODY PROBLEM

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ABSTRACT

The restricted four body problem, or equivalently the bicircular problem is defined. The Hamiltonian of the problem is constructed. The method of solution is briefly outlined. The solution of the problem is obtained using Delva-Hanslmeir perturbation technique. A computer program, using Mathematica 5.2, is built up to facilitate the lengthy cumbersome computation of the higher order perturbations.

Keywords: *Bicircular problem, Hamiltonian, Delva-Hanslmeir technique.*

1. INTRODUCTION

The well known Restricted Three Body Problem, e.g. the Earth-Moon system, (RTBP) defines the motion of a particle under the gravitational attraction of two massive bodies, usually called primaries, under the following assumptions:-

(i) The particle is so small that it does not affect the motion of the primaries,
(ii) The primaries are point masses that revolve in circular orbits around their common centre of mass.

It is usual to take a rotating reference frame with the origin at the centre of mass, and such that the two massive bodies are kept fixed on the x axis, the (x, y) plane is the plane of motion of the primaries, and the z axis is orthogonal to the (x, y) plane. These coordinates are sometimes called synodical.

1.1 Units Normalization or Non-dimensionalization

The system is made nondimensional by the following choice of units: the unit of distance is taken to be the constant separation between the centers of the two massive bodies, the unit of mass is the total mass of the primaries, their masses being μ and $1-\mu$ where $\mu \geq \frac{1}{2}$, and the unit of time is such that the period of less massive around the second massive body or more precisely around their common center of mass equals 2π. With this selection of units, it turns out that the universal constant of gravitation is also equal to one.

1.2 Lagrange's Equilibrium points

In this rotating coordinate system, the RTBP has five equilibrium points: three of them lie on the x axis (they are called collinear points, Eulerian points, or simply $L1, 2, 3$), and the other two form an equilateral triangle (in the (x, y) plane) with the primaries (they are called triangular points, Lagrangian points or simply $L4,5$). The collinear points are of the type centre×centre×saddle (for all μ), while the stability of the triangular points depends on the value of μ. If μ is lower than the Routh critical value $\mu_R = 3.8521 \times 10^{-2}$, these points are linearly stable. On the contrary, for $\mu_R < \mu < \frac{1}{2}$, these points are unstable. The usual cases in the solar system (like Sun-Jupiter or Earth-Moon) have a mass parameter lower than μ_R, so they are linearly stable; see Fig. 1. The details can be found in almost any textbooks on Celestial Mechanics (for instance, see Szebehely [1], Meyer and Hall [2] and Murray and Dermott [3]).

2. RESTRICTED FOUR-BODY MODELS: THE BICIRCULAR MODEL

Now, in this paper, we hope to introduce a concise Restricted Four Body Problem (RFBP) which can be used as an intermediate step for the exploration of the general, planner, or three dimensional N (≥ 4)-body problem. The RFBP is obtained by introducing an additional third massive body to the classical two primaries of the RTBP. Cronin et al. [4] included the effect of the Sun in the Earth-Moon RTBP. The simplest model of the RFBP is the Bicircular Restricted Problem. In this model we suppose that the Earth and Moon are revolving in circular orbits around their center of mass (barycenter) and the Earth-Moon barycenter is moving in a circular orbit around the center of mass of the Sun-Earth-Moon system. This model can be suitable to take into account the gravitational effect of the Sun in the Earth-Moon (RTBP), i.e. it can be considered as a periodic perturbation of the RTBP in which one primary has been split into two that move around their common center of mass. Then, once the motion of these three massive bodies has been prescribed (using very simple trigonometric expressions), it is not difficult to write the force acting on an infinitesimal particle and to derive the equations of motion for such a particle. It is usual to use the same reference frame as in the RTBP: the origin is taken at the Earth-Moon barycenter, with the same x-axis as in the RTBP. Hence, in these coordinates, the Sun is turning (clockwise) around the origin. For recent results on this model concerning the Earth-Moon-Sun case, see Celletti and Giorgilli [5], Gomez, et al. [6], Simo et al [7] and Jorba [8],[9].

The selection of circular trajectories for the three massive bodies is to avoid a too complex model, but unfortunately this situation leads to a breakdown of the coherent motion of these three massive bodies. That is, the assumed motions do not satisfy Newton's equations. Finally, using a suitable change of coordinates in which Earth and Moon are kept fixed on the x-axis, as in the RTBP, these equations of motion are written as a periodic time-dependent perturbation of the RTBP. In what follows, we will refer to these kinds of models as Bicircular Coherent Periodic Models or BCCP. A coherent model for the Earth-Moon-Sun problem has been developed in Andreu [10], and Andreu and Simo [11].

3. THE HAMILTONIAN OF A TEST PARTICLE IN THE RFBP

Assume that the two frames of reference, the inertial (ζ,η,ξ) and the synodic (x, y, z), coincide at $t = 0$. Let (ζ,η,ξ) and (x, y, z) be the position of P in the inertial and rotating frames, respectively. In normalized units, we have the following transformation of the particles position between the two frames:

$$\begin{pmatrix}\zeta\\ \eta\\ \xi\end{pmatrix} = M\begin{pmatrix}x\\ y\\ z\end{pmatrix}.$$

Where,

$$M = \begin{pmatrix}Cos\,t & -Sin\,t & 0\\ Sin\,t & Cos\,t & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Differentiating gives us the transformation of velocity components from the rotating to the inertial frame:

$$\begin{pmatrix}\dot\zeta\\ \dot\eta\\ \dot\xi\end{pmatrix} = \frac{dM}{dt}\begin{pmatrix}x\\ y\\ z\end{pmatrix} + M\begin{pmatrix}\dot x\\ \dot y\\ \dot z\end{pmatrix},$$

which can be written as:

$$\begin{pmatrix}\dot\zeta\\ \dot\eta\\ \dot\xi\end{pmatrix} = M\ J\begin{pmatrix}x\\ y\\ z\end{pmatrix} + M\begin{pmatrix}\dot x\\ \dot y\\ \dot z\end{pmatrix},$$

where,

$$J=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} \dot{\zeta} \\ \dot{\eta} \\ \dot{\xi} \end{pmatrix} = M \begin{pmatrix} \dot{x}-y \\ \dot{y}+x \\ \dot{z} \end{pmatrix}.$$

Recall the general form of the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}=0 ,$$

where the dynamical system is described by generalized coordinates ($\boldsymbol{q_1,....,q_n}$). One usually chooses the Lagrangian $\boldsymbol{L}$ to be of the form kinetic minus potential energy in the inertial frame of reference as

$$L\left(\zeta,\eta,\xi,\dot{\zeta},\dot{\eta},\dot{\xi};t\right)=\frac{1}{2}\left(\dot{\zeta}^2+\dot{\eta}^2+\dot{\xi}^2\right)-\Omega\left(\zeta,\eta,\xi\right) \tag{1}$$

In the rotating frame, the Lagrangian $\boldsymbol{L}$ is given by

$$L\left(x,y,z,\dot{x},\dot{y},\dot{z}\right)=\frac{1}{2}\left(\left(\dot{x}-y\right)^2+\left(\dot{y}+x\right)^2+\dot{z}\right)-\Omega\left(x,y,z\right) \tag{2}$$

where the potential-like function is given by,

$$\Omega\left(x,y,z\right)=\frac{\mu}{r_{P1}}+\frac{1-\mu}{r_{P2}}+m_{\odot}\left(\frac{1}{r_{P3}}-\frac{1}{r_{Sb}}\right)-\frac{1}{2}\mu\left(1-\mu\right) , \tag{3}$$

with,

$$M_0=-\frac{M_3}{r_{Sb}}-\frac{1}{2}\mu\left(1-\mu\right),\qquad M_1=\mu,\quad M_2=\left(1-\mu\right),\quad M_3=m_{\odot} ,$$

where $\boldsymbol{\mu}$ is the mass of the less massive body in dimensionless units, $\boldsymbol{m_{\odot}}$ is the mass of the Sun, and $\boldsymbol{r_{P1}},\boldsymbol{r_{P2}}$ and $\boldsymbol{r_{P3}}$ are given as:

$$r_{P1}=\sqrt{\left(x-\mu\right)^2+y^2+z^2} ,$$

$$r_{P2}=\sqrt{\left(x-\mu+1\right)^2+y^2+z^2} \text{, and,}$$

$$r_{P3}=\sqrt{\left(x-r_{Sb}\,Cos\left(\theta_0+n_{\odot}\,t\right)\right)^2+\left(y+r_{Sb}\,Sin\left(\theta_0+n_{\odot}\,t\right)\right)^2+z^2} ,$$

are the distances from the infinitesimally object to the Earth, to the Moon and to Sun respectively. and $\boldsymbol{r_{PS}}$ is the distance from the Sun to the Earth-Moon barycenter, $\boldsymbol{\theta_0}$ is the mean longitude of the Sun at an epoch, $\boldsymbol{n_{\odot}}$ is its mean motion. The constant last term in the expression for is added by convention (see, e.g., Llibre, Martinez and Simo [14]), and will not affect the equations of motion. For more illustration see Fig. 2.

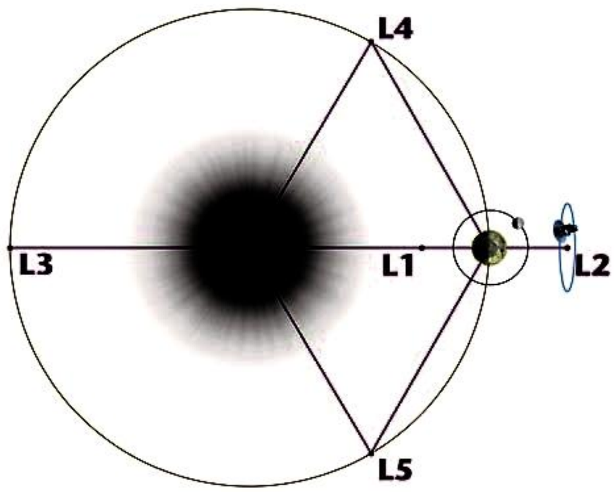

Fig.(1)

The theory of moving systems says that one can simply write down the Euler- Lagrange equations in the rotating frame and one will get the correct equations. It is a very efficient generic method for computing equations for either moving systems or for systems seen from moving frames. Whenever one has a Lagrangian system, one can transform it to Hamiltonian form by means of the Legendre transformation:

$$\begin{aligned} \boldsymbol{H}_{\boldsymbol{RFBP}}\left(q_i, p_i\right) &= \sum_{i=1}^{n} p_i\,\dot{q}_i - \boldsymbol{L}\left(q_i, p_i\right), \\ &= \frac{1}{2}\left(p_x^2+p_y^2+p_z^2\right)+\left(y\,p_x - x\,p_y\right)-\frac{1-\mu}{r_{P1}}-\frac{\mu}{r_{P2}}-\frac{m_{\square}}{r_{P3}}+\frac{m_{\square}}{r_{Sb}}. \qquad (4) \\ &\quad .\left(x\,Cos\left(\theta_0+n_{\square}\,t\right)-y\,Sin\left(\theta_0+n_{\square}\,t\right)\right), \end{aligned}$$

where the generalized momenta is given by : $\boldsymbol{p_i} = \dfrac{\partial \boldsymbol{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{i}}}$.

Notice that both the Lagrangian and the Hamiltonian form of the equations in rotating coordinates (***x, y***) give a time-independent system. Viewed as a dynamical system, it is a four dimensional dynamical system in either $\left(\boldsymbol{x},\boldsymbol{y},\dot{\boldsymbol{x}},\dot{\boldsymbol{y}}\right)$ or $\left(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p}_{\boldsymbol{x}},\boldsymbol{p}_{\boldsymbol{y}}\right)$ space.

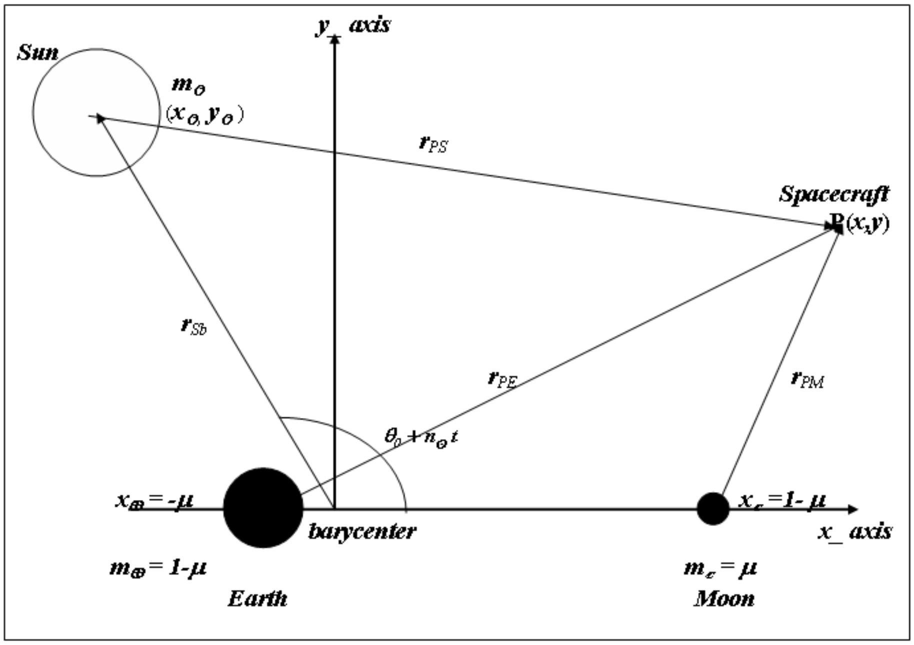

Fig. (2): In this figure the distance between the small body and the Earth, the Moon and the Sun are $r_{PE}\left(=r_1\right), r_{PM}\left(=r_2\right)$ and $r_{PS}\left(=r_3\right)$ respectively.

The numerical values of the included normalized parameters are as follows:

$\mu = 0.01215$ $m_{\square} = 328900.54$

$r_3 = 388.81114$ $n_{\square} = 0.925195985520347$

In computing the above numerical values, time is scaled by the period of the Earth and Moon around their center of mass ($\frac{T}{2\pi}$, where $T = 2.361 \times 10^6\ s$), positions are scaled by the average Earth-Moon distance ($r_{EM} = 3.850 \times 10^5\ km$), and velocities are scaled by the Moons average orbital speed around the Earth (= ***0.326 km/s***).

4. PERTURBATION APPROACH

In many cases in celestial mechanics, the series development of the disturbing function is not easily treated and is complicated. To avoid this difficulty we use an alternative approach developed independently by Delva and Hanslmeier, Delva [12] and Hanslmeier [13]. In this perturbative scheme the procedure can be performed with a special linear Lie operator which produces a Lie series. The convergence of the series is the same as for Taylor series, since the series is only another form of the Lie series. In addition we can change the step size easily (if necessary).

Let $H_{RFBP}\ (u_i,\ U_i,\ t)$ be the Hamiltonian function $u_i\left(=x,y,z\right)$ be the coordinates, $U_i\left(=p_x, p_y, p_z\right)$ be the momenta, and ***t*** be the time. Then the equations of motion are

$$\dot{u}_i = \frac{\partial H_{RFBP}}{\partial U_i} \qquad \dot{U}_i = -\frac{\partial H_{RFBP}}{\partial u_i}. \tag{5}$$

The linear Lie operator has the general form,

$$D = \sum_{i=0}\left(\frac{du_i}{dt}\frac{\partial}{\partial u_i} + \frac{dU_i}{dt}\frac{\partial}{\partial U_i} + \frac{\partial}{\partial t}\right) \quad . \tag{6}$$

The solution $\vec{u}_i\left(x, y, z, p_x, p_y, p_z, t\right)$, and $\vec{U}_i\left(x, y, z, p_x, p_y, p_z, t\right)$ are then given by the Lie series,

$$\vec{u}_i = \left[\left\{exp\left(t - t_0\right)D\right\}u_i\right]_{\vec{u}_i = \vec{u}_{i0}} = \sum_{j=0}\left[D^j\,\vec{u}_i\right]_{\vec{u}_{i0}}\frac{\left(t - t_0\right)^j}{j\,!} \; . \tag{7}$$

$$\vec{U}_i = \left[\left\{exp\left(t - t_0\right)D\right\}U_i\right]_{\vec{U}_i = \vec{U}_{i0}} = \sum_{j=0}\left[D^j\,\vec{U}_i\right]_{\vec{U}_{i0}}\frac{\left(t - t_0\right)^j}{j\,!} \quad , \tag{8}$$

where $D^j\,\vec{u}_i$, and $D^j\,\vec{U}_i$ are to be evaluated for the initial conditions $\vec{u}_{i0}$, and $\vec{U}_{i0}$.

5. SOLUTION OF THE PROBLEM

To obtain the general solution (7) and (8) of the concerned dynamical system of the RFBP we have the following successive application of the linear Lie operator (6) up to the third order $\vec{u}_i\left(= x, y, z\right)$ and $\vec{U}_i\left(= p_x, p_y, p_z\right)$.

$$\vec{u}_i = \sum_{j=0}\left[D^j\,\vec{u}_i\right]_{\vec{u}_{i0}}\frac{\left(t - t_0\right)^j}{j\,!}$$

$$= \sum_{j=0}\left[D^{j-1}\,\vec{U}_i\right]_{\vec{u}_{i0}}\frac{\left(t - t_0\right)^j}{j\,!}$$

and

$$\vec{U}_i = \sum_{j=0}\left[D^j\,\vec{U}_i\right]_{\vec{U}_{i0}}\frac{\left(t - t_0\right)^j}{j\,!}$$

$$= \sum_{j=0}\sum_{k=3(2)}^{2j+1}\sum_{l=1}^{3}\left[M_{jk}^{i0} + \frac{M_{jk}^{il}\left(u_i, U_i\right)}{r_{pl}^k}\right]_{\vec{U}_{i0}}\frac{\left(t - t_0\right)^j}{j\,!}$$

Where the involved non-vanishing coefficients are:

$$M_{13}^{10} = p_y - \frac{n_{\square}}{r_{Sb}}x_{\square}$$

$$M_{13}^{11} = -\frac{\left(1-\mu\right)\left(-\mu + x\right)}{r_1^3}$$

$$M_{13}^{12} = -\frac{\mu\left(1-\mu + x\right)}{r_2^3}$$

$$M_{13}^{13} = -\frac{m_{\square}\left(-x_{\square} + x\right)}{r_3^3}$$

$$M_{13}^{20} = -p_x - \frac{n_{\square}}{r_{Sb}} y_{\square}$$

$$M_{13}^{21} = -\frac{(1-\mu)y}{r_1^3}$$

$$M_{13}^{22} = -\frac{\mu y}{r_2^3}$$

$$M_{13}^{23} = -\frac{m_{\square}\left(-y_{\square} + y\right)}{r_3^3}$$

$$M_{13}^{31} = -\frac{(1-\mu)z}{r_1^3}$$

$$M_{13}^{32} = -\frac{\mu z}{r_2^3}$$

$$M_{13}^{33} = -\frac{m_{\square}\, z}{r_3^3}$$

$$M_{23}^{10} = -p_x - \frac{n_{\square}\left(1-n_{\square}\right)}{r_{Sb}} y_{\square}$$

$$M_{23}^{11} = -\frac{(1-\mu)}{r_1^3} p_x - \frac{(1-\mu)}{r_1^3} y$$

$$M_{23}^{12} = -\frac{\mu}{r_2^3} p_x - \frac{\mu}{r_2^3} y$$

$$M_{23}^{13} = \frac{m_{\square}\left(\left(n_{\square} - 1\right) y_{\square} + y\right)}{r_3^3} - \frac{m_{\square}}{r_3^3} p_x$$

$$M_{25}^{11} = \frac{3(1-\mu)(x-\mu)}{r_1^5}\left[(x-\mu)p_x + y\, p_y + z\, p_z\right]$$

$$M_{25}^{12} = \frac{3\mu(1+x-\mu)}{r_2^5}\left[(1+x-\mu)p_x + y\, p_y + z\, p_z\right]$$

$$M_{25}^{13} = \frac{3m_{\square}\left(x - x_{\square}\right)}{r_3^5}\left[n_{\square}\left(x_{\square}\, y - y_{\square}\, x\right) - \left(x - x_{\square}\right)p_x + \left(y - y_{\square}\right)p_y + z\, p_z\right]$$

$$M_{23}^{20} = \frac{-n_{\square}\left(n_{\square} - 1\right)}{r_{Sb}} x_{\square} - p_y$$

$$M_{23}^{21} = -\frac{(1-\mu)(x-\mu)}{r_1^3} p_x - \frac{(1-\mu)}{r_1^3} p_y$$

$$M_{25}^{22} = \frac{\mu(1+x-\mu)}{r_2^3} - \frac{\mu}{r_2^3} p_y$$

$$M_{23}^{23} = \frac{m_{\square}\left((n_{\square}-1)x_{\square}+x\right)}{r_3^3} - \frac{m_{\square}}{r_3^3} p_y$$

$$M_{25}^{21} = \frac{3(1-\mu)y}{r_1^5}\left[(x-\mu)p_x + y\ p_y + z\ p_z\right]$$

$$M_{25}^{22} = \frac{3\mu y}{r_2^5}\left[(1+x-\mu)p_x + y\ p_y + z\ p_z\right]$$

$$M_{25}^{23} = \frac{3m_{\square}(y-y_{\square})}{r_3^5}\left[n_{\square}(x_{\square}\ y - y_{\square}\ x) + (x-x_{\square})p_x + (y-y_{\square})p_y + z\ p_z\right]$$

$$M_{23}^{31} = -\frac{(1-\mu)}{r_1^3} p_z$$

$$M_{23}^{32} = -\frac{\mu}{r_2^3} p_z$$

$$M_{23}^{33} = -\frac{m_{\square}}{r_3^3} p_z$$

$$M_{25}^{31} = \frac{3(1-\mu)z}{r_1^5}\left[(x-\mu)p_x + y\ p_y + z\ p_z\right]$$

$$M_{25}^{32} = \frac{3\mu z}{r_2^5}\left[(1+x-\mu)p_x + y\ p_y + z\ p_z\right]$$

$$M_{25}^{33} = \frac{3m_{\square}\ z}{r_3^5}\left[n_{\square}(x_{\square}\ y - y_{\square}\ x) + (x-x_{\square})p_x + (y-y_{\square})p_y + z\ p_z\right]$$

6 . CONCLUSION AND OUTLOOK

The Hamiltonian and the canonical equations of motion of an infinitesimal body in the field of three massive gravitating bodies, ***RFBP,*** are formulated. An explicit perturbative solution using Delva-Hanslmeier technique is obtained by means of Lie series. The final results for the canonical solution are in explicit form for coordinates and conjugate momenta. To obtain the solution for higher orders a computer program code is designed using MATHEMATICA 5.2 software. Using this code we found the following recurrence relation $\boldsymbol{D_n\, u_i = D_{(n-1)}\, U_i}$, $\boldsymbol{u_i\left(= x, y, z\right)}$ & $\boldsymbol{U_i\left(= p_x, p_y, p_z\right)}$. This relation can save a considerable time when executing the program to higher order solutions.

The advantage of our option over the Von Zeipel procedure, is that the differential equations of motion are treated directly, avoiding the cumbersome reckoning work required in the determination of the Von Zeipel generating function and the subsequent formation of partial derivatives to express the implicit transformation

equations and the elimination process leading to explicit relations between the canonical sets of old and new variables.
We hope, in forthcoming studies, to compute the number, positions, and stability of the libration points in the RFBP. Also we aim to investigate the motion of an infinitesimal body near these fixed points. In addition to these points we search for more recurrence relations in our obtained solution.

REFERENCES

[1] V. Szebehely, ”Theory of Orbits”, Academic Press, 1967.

[2] K. R. Meyer and G. R. Hall, ”Introduction to Hamiltonian Dynamical Systems and the N-Body Problem” Springer, New York, 1992.

[3] Murray, C. D. and Dermott, S. F. (1999): ”Solar system dynamics”, Cambridge university press

[4] J. Cronin, P.B. Richards, and L.H. Russell, ”Some periodic solutions of a fourbody problem”, Icarus, 3:423428, 1964.

[5] A. Celletti and A. Giorgilli, ”On the stability of the Lagrangian points in the spatial Restricted Three Body Problem”, Celestial Mech., 50(1):3158, 1991.

[6] G. Gomez, ‘A. Jorba, J. Masdemont, and C. Simo, ”Study of Poincare maps for orbits near Lagrangian points”, ESOC contract 9711/91/D/IM(SC), final report, European Space Agency, 1993.

[7] C. Simo, G. Gomez, ‘A. Jorba, and J. Masdemont, ”The Bicircular model near the triangular libration points of the RTBP”, In A.E. Roy and B.A. Steves, editors, From Newton to Chaos, pages 343370, New York, 1995. Plenum Press.

[8] A. Jorba, ”A numerical study on the existence of stable motions near the triangular points of the real Earth-Moon system”, Astron. Astrophys., 364(1):327338, 2000.

[9] A. Jorba, ”On practical stability regions for the motion of a small particle close to the equilateral points of the real Earth-Moon system”, Proceedings of the III International Symposium HAMSYS-98. Geometrical Methods in Hamiltonian Systems and Celestial Mechanics. P´atzcuaro (M´exico), December 711, 1998. World Scientific (to appear), 2000.

[10] M.A. Andreu, ”The quasi-bicircular problem”, PhD thesis, Univ. Barcelona, 1998.

[11] M.A. Andreu and C. Simo, ”The quasi-bicircular problem for the Earth-Moon-Sun parameters” Preprint, 2000.

[12] Delva, M. (1984) : Celest. Mech.. 34, 145.

[13] Hanslmeier, A. (1984) : Celest. Mech. 34, 135.

[14] Llibre, J., R. Martinez and C. Simo (1985), ”Transversality of the invariant manifolds associated to the Lyapunov family of periodic orbits near L2 in the restricted three-body problem”, J. Diff. Eqns. 58, 104-156.