THE APPROXIMATE AND EXACT SOLUTIONS OF THE SPACE- AND TIME-FRACTIONAL BURGERS EQUATIONS

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ABSTRACT

In the paper, we extend the differential transform method to solve nonlinear fractional partial differential equations. The time- and space-fractional Burgers equations with initial conditions are chosen to illustrate our method. As a result, we successfully obtain some available approximate solutions of them. The results reveal that the proposed method is very effective and simple for obtaining approximate solutions of nonlinear fractional partial differential equations. The fractional derivatives are considered in the Caputo sense.

Keywords: *Time- and space-fractional Burgers equations; Fractional derivative; Differential transform method*
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1. INTRODUCTION

In the last past decades, nonlinear fractional partial differential equations are widely used to describe many important phenomena and dynamic processes in physics, such as engineering, electromagnetics, acoustics, viscoelasticity, electrochemistryand material science [1–4]. For better understanding the phenomena that a given nonlinear fractional partial differential equation describes, the solutions of differential equations of fractional order is much involved. In general, there exists no method that yields an exact solution for nonlinear fractional partial differential equations.

The fractional differential equations (FDE) appear more and more frequently in different research areas and engineering applications. Most recently, Momani [5] has presented nonperturbative analytical solutions of the space- and time-fractional Burgers equations by Adomian decomposition method. Inc [6] used variational iteration method for solving space- and time-fractional Burgers equations. Wang [7,8] extend the application of the homotopy perturbation and Adomian decomposition methods to construct approximate solutions for the nonlinear fractional KdV-Burgers equation. The space-fractional Burgers equation describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative results from the memory effect of the wall friction through the boundary layer. The same form can be found in other systems such as shallow-water waves and waves in bubbly liquids. For more details on the applications associated with of the space-fractional Burgers equation [9].

We consider non-perturbation analytical solutions of the generalized Burgers equation with time- and space-fractional derivatives of the form[5]:

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\eta\frac{\partial^\beta u}{\partial x^\beta}=v\frac{\partial^2 u}{\partial x^2}-\varepsilon u\frac{\partial u}{\partial x},\ t>0,\ 0<\alpha,\beta\le 1, \qquad (1.1)$$

where ε, v and η are parameters and $\beta=0$ and $\alpha=0$ are parameters describing the order of the fractional time- and space-derivatives, respectively. The function $u(x,t)$ is assumed to be a causal function of time and space, i.e., vanishing for $t<0$ and $x<0$. The fractional derivatives are considered in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. We refer to Eq. (1.1) as to the time-fractional Burgers and to the space-fractional Burgers equation in the cases $\{0<\alpha\le 1,\ \eta=0\}$ and $\{0<\beta\le 1,\ \alpha=1\}$, respectively.

The DTM was first applied in the engineering domain in [10]. Hashim [11] demonstrated the application of homotopy-perturbation method for solving fmKdV. Kurulay [12] applied the application of DTM method for solving fmKdV.

The paper is organized as follows. A brief review of the fractional calculus theory is given. We use the Differential transform method to construct our exact solutions of the space- and time-fractional Burgers equations. We present two examples to show the efficiency and simplicity of the proposed method. Conclusions will be presented in final.

2. BASIC DEFINITIONS

In this section, let us recall essentials of fractional calculus first. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. We have well known definitions of a fractional derivative of order $\alpha > 0$ such as Riemann–Liouville, Grunwald–Letnikow, Caputo and Generalized Functions Approach [1, 4]. The most commonly used definitions are the Riemann–Liouville and Caputo. For the purpose of this paper the Caputo's definition of fractional differentiation will be used, taking the advantage of Caputo's approach that the initial conditions for fractional differential equations with Caputo's derivatives take on the traditional form as for integer-order differential equations. We give some basic definitions and properties of the fractional calculus theory which were used through paper.

Definition 2.1. A real function $f(x), x > 0,$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $(p > \mu)$, such that $f(x) = x^p f_1(x),$ where $f_1(x) \in C[0,\infty),$ and it said to be in the space C_μ^m iff $f^m \in C_\mu, m \in N.$

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0,$ of a function $f \in C_\mu, \mu \geq -1,$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \left(x-t\right)^{\alpha-1} f(t)dt, \quad \alpha > 0,$$

$$J^0 f(x) = f(x).$$

It has the following properties:

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > 1$:

$$1. J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x),$$

$$2. J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x),$$

$$3. J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

The Riemann–Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for the physical problems of the real world since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

Definition 2.3. The fractional derivative of $f(x)$ in the caputo sense is defined as

$$D_*^\nu f(x) = J^{m-\nu} D^m f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x (x-t)^{m-\nu-1} f^{(m)}(t) dt,$$

for $m-1 < \nu < m,\ m \in N,\ x > 0,\ f \in C_{-1}^m.$

Lemma 2.1. If $m-1 < \alpha < m,\ m \in N$ and $f \in C_\mu^m, \mu \geq -1,$ then

$$D_*^{\alpha} J^{\alpha} f(x) = f(x),$$

$$J^{\alpha} D_*^{v} f(x) = f(x) - \sum_{k=0}^{m-1} f^{k}(0^{+}) \frac{x^{k}}{k!}, \text{ x>0.}$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

Definition 2.4. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_{*t}^{\alpha} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\xi)^{m-\alpha-1} \frac{\partial^m u(x,\xi)}{\partial \xi^m} d\xi, & \text{for } m-1 < \alpha < m, \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in N \end{cases}$$

and the space-fractional derivative operator of order $\beta > 0$ is defined as

$$D_{*x}^{\alpha} u(x,t) = \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} = \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^x (x-\theta)^{m-\beta-1} \frac{\partial^m u(\theta,t)}{\partial \theta^m} d\theta, & \text{for } m-1 < \beta < m, \\ \dfrac{\partial^m u(x,t)}{\partial x^m}, & \text{for } \beta = m \in N. \end{cases}$$

3. DIFFERENTIAL TRANSFORM METHOD

The DTM is applied to the solution of electric circuit problems. The DTM is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the DTM obtains a polynomial series solution by means of an iterative procedure. The method is well addressed in [13].

Consider a function of two variables $u(x,y),$ and suppose that it can be represented as a product of two single-variable functions, i.e., $u(x,y) = f(x)g(y)$. Based on the properties of generalized two-dimensional differential transform [14,15], the function $u(x,y)$ can be represented as

$$\begin{aligned} u(x,y) &= \sum_{k=0}^{\infty} F_{\alpha}(k)(x-x_0)^{k\alpha} \sum_{h=0}^{\infty} G_{\beta}(h)(y-y_0)^{h\beta} \\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta} \end{aligned} \tag{3.1}$$

where $0 < \alpha, \beta \le 1, U_{\alpha\beta}(k,h) = F_{\alpha}(k) G_{\beta}(h)$ is called the spectrum of $u(x,y)$. The generalized two-dimensional differential transform of the function $u(x,y)$ is given by

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)} \left[(D_{*x_0}^{\alpha})^{k} (D_{*y_0}^{\beta})^{h} u(x,y) \right]_{(x_0,y_0)}, \tag{3.2}$$

where $\left(D_{*x_0}^{\alpha}\right)^{k} = D_{*x_0}^{\alpha} D_{*x_0}^{\alpha} \cdots D_{*x_0}^{\alpha}, \cdots k\text{-times.}$ In case of $\alpha = 1$ and $\beta = 1$ the generalized two-dimensional differential transform (3.1) reduces to the classical two-dimensional differential transform.[16].

The operators in two-dimensional differential transformation Method [16]:

Let $U_{\alpha,\beta}(k,h), V_{\alpha,\beta}(k,h)$ and $W_{\alpha,\beta}(k,h)$ be the differential transformations of the functions $u(x,y)$, $v(x,y)$ and $w(x,y)$:

(a) If $u(x,y)=v(x,y)\pm w(x,y),$ then $U_{\alpha,\beta}(k,h)=V_{\alpha,\beta}(k,h)\pm W_{\alpha,\beta}(k,h)$,

(b) If $u(x,y)=av(x,y), a\in R,$ then $U_{\alpha,\beta}(k,h)=aV_{\alpha,\beta}(k,h)$,

(c) If $u(x,y)=v(x,y)w(x,y),$ then $U_{\alpha,\beta}(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}V_{\alpha,\beta}(r,h-s)W_{\alpha,\beta}(k-r,s),$

(d) If $u(x,y)=(x-x_0)^{n\alpha}(y-y_0)^{m\beta},$ then $U_{\alpha,\beta}(k,h)=\delta(k-n)\delta(h-m),$

(e) If $u(x,y)=v(x,y)w(x,y)q(x,y),$ then

$$U_{\alpha,\beta}(k,h)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{t=0}^{h}V_{\alpha,\beta}(r,h-s-p)W_{\alpha,\beta}(t,s)Q_{\alpha,\beta}(k-r-t,p),$$

(f) If $u(x,y)=D_{x_0}^{\alpha}v(x,y), 0<\alpha\le 1,$ then $U_{\alpha,\beta}(k,h)=\dfrac{\Gamma\left(\alpha(k+1)+1\right)}{\Gamma\left(\alpha k+1\right)}V_{\alpha,\beta}(k+1,h),$

(g) If $u(x,y)=f(x)g(y)$ and the function $f(x)=x^{\lambda}h(x),$ where $\lambda>-1, h(x)$ has the generalized Taylor series expansion $h(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^{\alpha k},$ and [16],

(i). $\beta<\lambda+1$ and α arbitrary or

(ii). $\beta\ge\lambda+1,\alpha$ arbitrary and $a_n=0$ for $n=0,1,...m-1,$ where $m-1<\beta\le m.$

Then the generalized differential transform (3.2) becomes

$$U_{\alpha,\beta}(k,h)=\frac{1}{\Gamma\left(\alpha k+1\right)\Gamma\left(\beta h+1\right)}\left[D_{*x_0}^{\alpha k}(D_{*y_0}^{\beta})^{h}u(x,y)\right]_{(x_0,y_0)},$$

(h) If $u(x,y)=D_{x_0}^{\gamma}v(x,y), m-1<\gamma\le m$ and $v(x,y)=f(x)g(y),$ then

$$U_{\alpha,\beta}(k,h)=\frac{\Gamma\left(\alpha k+\gamma+1\right)}{\Gamma\left(\alpha k+1\right)}V_{\alpha,\beta}(k+\gamma/\alpha,h).$$

(i) If $u(x,y,t)=D_{*x_0}^{\alpha}v(x,y,t),\ 0<\alpha\le 1$ then

$$U_{\alpha,\beta,\gamma}(k,h,m)=\frac{\Gamma\left(\alpha(k+1)+1\right)}{\Gamma\left(\alpha k+1\right)}V_{\alpha,\beta,m}(k+1,h,m).$$

(j) If $u(x.y)=a(x,y)\dfrac{\partial v(x,y)}{\partial x}$ then

$$U(k,h)=\sum_{i=0}^{k}\sum_{j=0}^{h}(k-i+1)A(i,j)U(k-i+1,h-j).$$

The proofs of the some properties can be found in [16].

4. APPLICATIONS

In order to illustrate the advantages and the accuracy of the DTM for solving nonlinear fractional Burgers equation, we have applied the method to two different examples. In the first example, we consider a nonlinear time-fractional Burgers equation, while in the second example, we consider a nonlinear space-fractional equation. All the results are calculated by using the symbolic computation software Maple.

4.1. Approximate solution of time-fractional Burgers equation

we consider the following time-fractional Burgers equation [5]:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}-v\frac{\partial^{2}u}{\partial x^{2}}+\varepsilon u\frac{\partial u}{\partial x}=0,\quad 0<\alpha\le 1,\ \ t>0. \qquad (4.1)$$

We consider Eq. (4.1) with $\varepsilon = 1$ and the following initial condition [17]:

$$u(x,0) = g(x) = \frac{\mu + \sigma + (\sigma - \mu)\exp(\gamma)}{1 + \exp(\gamma)}, \ t \geq 0, \tag{4.2}$$

where $\gamma = \frac{\mu}{v}(x - \lambda)$ and the parameters μ, σ, λ and v are arbitrary constant.

Taking the differential transform of Eq. (4.1) by using the related property, we have

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_\alpha(k,h+1) + \varepsilon \sum_{i=0}^{k} \sum_{j=0}^{h} (k-i+1) U_\alpha(k-i+1, j) U_\alpha(i, h-j) = v(k+2)(k+1) U_\alpha(k+2, h).$$

We start with an initial conditions, that was given by Eq. (4.2). The solution for the fractional Burgers Eq. (4.1) in a series form is given by

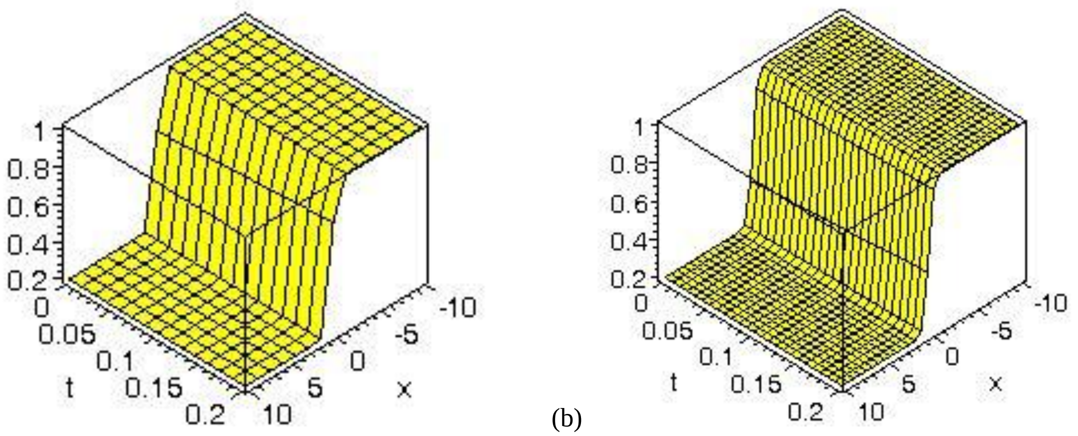
$$u(x,t) = \frac{\mu + \sigma + (\sigma - \mu)\exp(\gamma)}{1 + \exp(\gamma)} + [vg'' - \varepsilon g g'] \frac{t^\alpha}{\Gamma(\alpha+1)}$$
$$+ [2\varepsilon^2 g (g')^2 + \varepsilon^2 g^2 g'' - 4\varepsilon v g' g'' - 2\varepsilon v g g'' + v^2 g^{(4)}] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots.$$

Thus, we have the solution of (4.1) in a series form for $\alpha = 1$,

$$u(x,t) = \frac{\mu + \sigma + (\sigma - \mu)\exp(\gamma)}{1 + \exp(\gamma)} + \frac{2\mu\sigma^2 \exp(\gamma)}{[1+\exp(\gamma)]^2 v} t + \frac{[\mu^3\sigma^2 \exp(\gamma)][\exp(\gamma) - 1]}{[1+\exp(\gamma)]^3 v^2} t^2$$
$$+ \frac{[\mu^4\sigma^3 \exp(\gamma)][1 - 4\exp(\gamma) + \exp(\gamma)]}{3[1+\exp(\gamma)]^4 v^3} t^3 + \cdots,$$

$$u(x,t) = \frac{\mu + \sigma + (\sigma - \mu)\exp[\frac{\mu}{v}(x - \sigma t - \lambda)]}{1 + \exp[\frac{\mu}{v}(x - \sigma t - \lambda)]},$$

which are exactly same as obtained by Adomian decomposition [5] and variational iteration methods [6].



(a) (b)

Fig. 1. The surface shows the solution $u(x,t)$ for Eq. (4.1): (a) exact solution; (b)approximate solution when $\alpha = 1, \ v = 0.1 \ \mu = 0.4 \ \sigma = 0.6 \ \lambda = 0.125$ and $\varepsilon = 1.$

4.2. Approximate solution of space-fractional Burgers equation

In this section, we consider the following space-fractional Burgers equation [5]:

$$\frac{\partial u}{\partial t}+\varepsilon u\frac{\partial u}{\partial x}-v\frac{\partial^2 u}{\partial x^2}+\eta\frac{\partial^\beta u}{\partial x^\beta}=0,\quad 0<\beta\le 1,\quad \mathrm{x},\, t>0, \tag{4.3}$$

subject to the initial conditions

$$u(0,t)=0,\ u_x(0,t)=\frac{1}{t}-\frac{\pi^2}{2vt^2}. \tag{4.4}$$

Taking the differential transform of Eq. (4.3), we have

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}\eta U_\beta(k+1,h)=v(k+2)(k+1)U_\beta(k+2,h)$$
$$-\varepsilon\sum_{i=0}^{k}\sum_{j=0}^{h}(k-i+1)U_\beta(k-i+1,j)U_\beta(i,h-j)-(h+1)U_\beta(k,h+1). \tag{4.5}$$

For purposes of illustration of the differential transform method for solving Burgers equation with space-fractional derivative, consider (4.3) with $\varepsilon=1$ and subject to the initial conditions

$$u(0,t)=0,\ u_x(0,t)=\frac{1}{t}-\frac{\pi^2}{2vt^2},\ u(x,1)=x-\pi\tanh\left[\frac{\pi x}{2v}\right]. \tag{4.6}$$

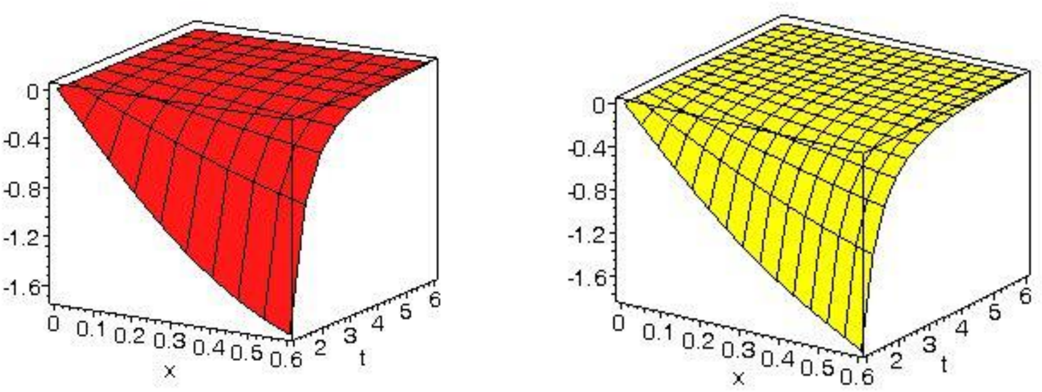
We substitute the initial conditions (4.6) into (4.5), for the special case $\beta=1$, we obtain

$$u(x,t)=\left(\frac{1}{t}-\frac{\pi^2}{2vt^2}\right)x+\frac{\eta x^2}{2v}+\left(\frac{\eta}{6v}+\frac{\pi^4}{24v^3t^4}\right)x^3+\frac{\eta\left(-\pi^2+2vt\right)x^4}{16v^3t^2}-\frac{\pi^6x^5}{240v^5t^6}+\cdots$$

which is the solution of (4.3) in series form.

The exact solution [5], for the special case $\eta=0$, is given by

$$u(x,t)=\frac{x}{t}-\frac{\pi}{t}\tanh\left[\frac{\pi x}{2vt}\right].$$



(a) (b)

Fig. 2. The surface shows the solution $u(x,t)$ for Eq. (4.3): (a) exact solution; (b)approximate solution. The parameters have the following values $\eta=0,\ v=1$ and $\varepsilon=1.$

5. CONCLUSIONS

The fundamental goal of this work has been to obtain analytical solutions of the space- and time fractional Burgers equations. This goal has been achieved by using differential transform method and an approximation series solution can be obtained to any desired number of terms.

We extend the differential transform method to solve nonlinear fractional partial differential equations. The time- and space-fractional Burgers equation with initial conditions is chosen to illustrate the proposed method. As results, based on symbolic computation system Maple, some approximate solutions of fractional Burgers equation with high accuracy are obtained. The obtained results demonstrate the reliability of the algorithm and its wider applicability to nonlinear fractional partial differential equations. We hope other traditional analytic methods for nonlinear differential equations of integer order can be extended to nonlinear fractional calculus equations and this will be investigated in following work.

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