A COMPARISON OF POWERS OF CONDITIONAL AND UNCONDITIONAL TEST USING A POISSON DISTRIBUTION

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ABSTRACT
This paper examines the powers of conditional and unconditional test using a Poisson distribution and determination of sample sizes. It was observed that equal sample sizes of 33 of the two populations in conditional test gives the same power with that of unconditional test for sample size 27 of population 1 and sample size 36 of population 2.

Keywords: Powers, Conditional, Unconditional exact, indicator function, two-tail, Type I error

1. INTRODUCTION
The exact powers of the test can be computed using Poisson probabilities and an indicator function. For instance, for a given \( \lambda \) and \( \lambda_0 \), the power of the test for hypothesis \( H_0: \lambda \leq \lambda_0 \) versus \( H_\alpha: \lambda > \lambda_0 \) can be computed using the test-statistic

\[
\sum_{k=0}^{\infty} e^{-\lambda_0} \left( \frac{\lambda}{\lambda_0} \right)^k \frac{1}{k!} \left[ P(K \geq k/n\lambda_0) \leq \alpha \right] \quad \hdots \quad (1)
\]

where \( k \sim \text{Poisson} \left( n\lambda_0 \right) \). Powers of the right-tail test and two-tail test can be expressed similarly.

1.1 Powers of the conditional test:
For given sample sizes, guess values of the means and a level of significance the exact power of the conditional test

\[
H_0: \frac{\lambda_1}{\lambda_2} \leq C \quad \text{against} \quad H_\alpha: \frac{\lambda_1}{\lambda_2} > C, \quad \hdots \quad (2)
\]

where \( C \) is a given positive number. The \( p \)-value based on the conditional distribution of \( k \), given \( k_1 + k_2 = m \) is given by

\[
P(k_1 \geq k/m, p) = \sum_{x=k}^{m} \binom{m}{x} P' \left( (1-p)^{m-x} \right) \quad \hdots \quad (3)
\]

where \( p = \frac{n_1 c / n_2}{1 + n_1 c / n_2} \)

The conditional test rejects the null hypothesis whenever the \( p \)-value is less than or equal to the specified nominal \( \alpha \) (Chapman 1952).

Also, the \( p \)-value of a left-tail test or a two-tail test can be expressed similarly.

Furthermore, the conditional test in (2) above can be calculated by the expression stated below

\[
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-n_1 \lambda_1}}{i!} \cdot \frac{e^{-n_2 \lambda_2}}{j!} I[P(X_i \geq i + j, p) \leq \alpha] \quad \hdots \quad (4)
\]

where \( I \) is the indicator, and \( P(X_i \geq k/m,p) \) and \( p \) are as defined in (3).

The powers of a two-tail test and left-tail test can be expressed similarly.

1.2 Empirical Data/Illustration
Suppose that a researcher hypothesizes that the mean \( \lambda_1 = 3 \) of a Poisson population is 1.5 times larger than the mean \( \lambda_2 \) or another population, and he likes to test

\[
H_0: \frac{\lambda_1}{\lambda_2} \leq 1.5 \quad \text{versus} \quad H_\alpha: \frac{\lambda_1}{\lambda_2} > 1.5
\]

To find the required sample size to get a power of 0.80 at the level 0.05, set 30 for both sample sizes,1 for one-tail test, 0.05 for level, 3 for (guess \( m_1 \)), 2 for (guess \( m_2 \)), the power is 0.76827. By increasing the sample size to 33, we...
get a power of 0.804721. Furthermore, when both sample sizes are 32, power is 0.793161. Therefore, the required sample size is 33.

The above result is achieved by trial and error of sample size.

1.3 Powers of the unconditional test

For a given \( \hat{\lambda}_1, \lambda_2 \), and a level of significance \( \alpha \), the power of the unconditional test is given by

\[
\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} e^{-\frac{n_1 \lambda_1}{k_1}} \frac{(n_1 \lambda_1)^{k_1}}{k_1!} e^{-\frac{n_2 \lambda_2}{k_2}} \frac{(n_2 \lambda_2)^{k_2}}{k_2!} \mathbb{P}(K_1, K_2) \leq \alpha.
\]

where \( p(k_1, k_2) \) is the p-value given by

\[
p[k_1, k_2] = \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\infty} e^{-\eta} \frac{(\eta)^{x_1}}{x_1!} e^{-\delta} \frac{(\delta)^{x_2}}{x_2!} \mathbb{I}[Z(x_1, x_2) \geq Z(k_1, k_2)].
\]

where \( \eta = n_1 (\hat{\lambda} + d), \delta = n_2 \hat{\lambda}d \) and

\[
Z(x_1, x_2) = \frac{x_1 - x_2 - d}{\sqrt{x_1 + x_2}}.
\]

and

\[
Z(k_1, k_2) = Z(x_1, x_2) \text{ with } x \text{ replaced by } k.
\]

It should be noted that, the null hypothesis will be rejected whenever the p-value is less than or equal to the nominal level \( \alpha \).

When \( \lambda_1 = \lambda_2 \), the equation (5) gives the size (i.e; actual Type I error rate) of the test.

The above formula will be used to compute the power of the test for the difference between two means.

1.4 Illustration

Suppose a researcher hypothesizes that the mean \( \lambda_1 = 3 \) of a Poisson population is at least one unit must larger than the mean \( \lambda_2 \) of another population, and he likes to test

\[
H_0 : \lambda_1 - \lambda_2 \leq 0 \text{ versus } H_\alpha : \lambda_1 - \lambda_2 > 0.
\]

In finding the required sample size to get a power of 0.80 at level of 0.05, set 30 for both sample sizes, 0 for \( d \) in \( [H_0; m_1 - m_2 = d] \), 1 for one-tail test, 0.05 for level, 3 for \( [\text{Guess } m_1] \), 2 for \( (\text{guess } m_2) \), to get power 0.791813. By increasing the sample size to 31, we also note that when the first sample size is 27 and the second sample size is 36, the power is 0.803128.

1.5 Observation:

In this paper, we observed that equal sample sizes of 33 for population one and two for conditional test gives a power that is the same with sample sizes of 27 for population 1 and 36 for population 2 unconditional test

1.6 Result

The result obtained in the above test of powers of the conditional and unconditional test reveals that the sample size of 33 gives a power of 0.804721 for conditional test while the sample sizes of first sample size 27 and the second sample size is 36 gives a power of 0.803128.

<table>
<thead>
<tr>
<th>Test</th>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>Power</th>
<th>Level of Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conditional</td>
<td>33</td>
<td>33</td>
<td>*0.804721</td>
<td>( \alpha = 0.05 )</td>
</tr>
<tr>
<td>Unconditional</td>
<td>27</td>
<td>36</td>
<td>*0.803128</td>
<td>( \alpha = 0.05 )</td>
</tr>
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<td>Conditional</td>
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<td>30</td>
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<td>( \alpha = 0.05 )</td>
</tr>
<tr>
<td>Unconditional</td>
<td>30</td>
<td>30</td>
<td>0.791813</td>
<td>( \alpha = 0.05 )</td>
</tr>
</tbody>
</table>

* = Power of the test.

REFERENCES
