FIXED POINT THEOREM IN MENGER SPACE FOR SEMI-COMPATIBLE MAPPINGS

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ABSTRACT
In this paper, the concept of semi-compatibility and weak compatibility in Menger space has been applied to prove a common fixed point theorem for six self maps.

Keywords: Probabilistic metric space, Menger space, common fixed point, compatible maps, semi-compatible maps, weak compatibility.

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1. INTRODUCTION
There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [4]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [8] studied this concept and gave some fundamental results on this space. Sehgal and Bharucha-Reid [9] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed-point theory in Menger space.

Recently, Jungck and Rhoades [3] termed a pair of self maps to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. Sessa [10] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weak commuting maps in metric spaces. Jungck [2] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [5].


In this paper a fixed point theorem for six self maps has been proved using the concept of semi-compatible maps and weak compatible maps.

2. PRELIMINARIES
Definition 2.1. A mapping $f : R \rightarrow R^+$ is called a distribution if it is non-decreasing left continuous with
$$\inf \{ f(t) \mid t \in R \} = 0 \quad \text{and} \quad \sup \{ f(t) \mid t \in R \} = 1.$$ We shall denote by $L$ the set of all distribution functions while $H$ will always denote the specific distribution function defined by
$$H(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases}.$$ 

Definition 2.2. A triangular norm * (shortly t-norm) is a binary operation on the unit interval $[0, 1]$ such that for all $a, b, c, d \in [0, 1]$ the following conditions are satisfied:
(a) $a * 1 = a$;
(b) $a * b = b * a$;
(c) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;
(d) $a * (b * c) = (a * b) * c$.

Examples of t-norms are $a * b = \max \{a + b - 1, 0\}$ and $a * b = \min \{a, b\}$.

Definition 2.3. [8] A probabilistic metric space (PM-space) is an ordered pair $(X, f)$ consisting of a non-empty set $X$ and a function $f : X \times X \rightarrow L$, where $L$ is the collection of all distribution functions and the value of $f$ at $(u, v) \in X \times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ assumed to satisfy the following conditions:
(PM-1) $F_{u,v}(x) = 1$, for all $x > 0$, if and only if $u = v$;
(PM-2) \( F_{x,x}(0) = 0; \)
(PM-3) \( F_{x,y} = F_{y,x}; \)
(PM-4) If \( F_{x,y}(x) = 1 \) and \( F_{v,w}(y) = 1 \) then \( F_{u,w}(x + y) = 1, \)
\( \text{for all } u,v,w \in X \text{ and } x, y > 0. \)

**Definition 2.4.** [8] A Menger space is a triplet \((X, \mathcal{F}, *)\) where \((X, \mathcal{F})\) is a PM-space and \( * \) is a t-norm such that the inequality

\( (PM-5) \quad F_{u,w}(x + y) \geq F_{u,v}(x) * F_{v,w}(y), \quad \text{for all } u, v, w \in X, x, y \geq 0. \)

**Proposition 2.1.** [9] Let \((X, d)\) be a metric space. Then the metric \(d\) induces a distribution function \( F \) defined by \( F_{x,y}(\varepsilon) = H(\varepsilon - d(x,y)) \) for all \( x, y \in X \) and \( \varepsilon > 0. \) If \( t \)-norm \( * \) is a \( * b = \min\{a, b\} \) for all \( a, b \in [0, 1] \) then \((X, F, *)\) is a Menger space. Further, \((X, F, *)\) is a complete Menger space if \((X, d)\) is complete.

**Definition 2.5.** [5] Let \((X, F, *)\) be a Menger space and \( * \) be a continuous t-norm.
(a) A sequence \( \{x_n\} \) in \( X \) is said to be converge to a point \( x \) in \( S \) (written \( x_n \to x \)) iff for every \( \varepsilon > 0 \) and \( \lambda \in (0,1) \), there exists an integer \( n_0 = n_0(\varepsilon, \lambda) \) such that \( F_{x,x}(\varepsilon) > 1 - \lambda \) for all \( n \geq n_0. \)
(b) A sequence \( \{x_n\} \) in \( X \) is said to be Cauchy if for every \( \varepsilon > 0 \) and \( \lambda \in (0,1) \), there exists an integer \( n_0 = n_0(\varepsilon, \lambda) \) such that \( F_{x,x}(\varepsilon) > 1 - \lambda \) for all \( n \geq n_0 \) and \( p > 0. \)
(c) A Menger space in which every Cauchy sequence is convergent is said to be complete.

**Remark 2.1.** If \( * \) is a continuous t-norm, it follows from \((PM-4)\) that the limit of sequence in Menger space is uniquely determined.

**Definition 2.6.** [11] Self mappings \( A \) and \( S \) of a Menger space \((X, F, t)\) are said to be weak compatible if they commute at their coincidence points i.e. \( Ax = Sx \) for \( x \in X \) implies \( ASx = SAx. \)

**Definition 2.7.** [5] Self mappings \( A \) and \( S \) of a Menger space \((X, F, t)\) are said to be compatible if \( F_{ASx_n} \to 1 \) for all \( x > 0, \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( Ax_n, Sx_n \to u \) for some \( u \) in \( X, \) as \( n \to \infty. \)

**Definition 2.8.** Self mappings \( A \) and \( S \) of a Menger space \((X, F, t)\) are said to be semi-compatible if \( F_{ASx_n} \to 1 \) for all \( x > 0, \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( Ax_n, Sx_n \to u, \) for some \( u \) in \( X, \) as \( n \to \infty. \)

Now, we give an example of pair of self maps \((S, T)\) which is semi-compatible but not compatible. Further we observe here that the pair \((T, S)\) is not semi-compatible though \((S, T)\) is semi-compatible.

**Example 2.1.** Let \((X, d)\) be a metric space where \( X = [0, 1] \) and \((X, F, t)\) be the induced Menger space with \( F_{p,q}(\varepsilon) = H(\varepsilon - d(p, q)), \forall p, q \in X \) and \( \forall \varepsilon > 0. \) Define self maps \( S \) and \( T \) as follows:

\[
Sx = \begin{cases}
x & \text{if } 0 \leq x < \frac{1}{2} \\
1 & \text{if } \frac{1}{2} \leq x \leq 1
\end{cases}
\]

\[
Tx = \begin{cases}
1 & \text{if } 0 \leq x < \frac{1}{2} \\
\frac{1}{2} & \text{if } \frac{1}{2} \leq x \leq 1
\end{cases}
\]

Take \( x_n = \frac{1}{2} - \frac{1}{n}. \) Now,

\[
F_{Sx_n,1/2}(\varepsilon) = H(\varepsilon - (1/n)).
\]

Therefore, \( \lim_{n \to \infty} F_{Sx_n,1/2}(\varepsilon) = H(\varepsilon) = 1. \)

Hence, \( Sx_n \to 1/2 \) as \( n \to \infty. \)

Similarly, \( Tx_n \to 1/2 \) as \( n \to \infty. \)

Also
\[ F_{STx_nT^n}(\epsilon) = H\left( \epsilon - \frac{1}{2} \right) \neq 1, \quad \forall \epsilon > 0. \]

Hence, the pair \((S, T)\) is not compatible.

Again, \( \lim_{n \to \infty} F_{STx_nT^n}(\epsilon) = \lim_{n \to \infty} F_{STx_n}(\epsilon) = H(\epsilon - 1) = 1 \forall \epsilon > 0. \)

Thus, \((S, T)\) is semi-compatible.

Now, \( \lim_{n \to \infty} F_{STx_nS^n}(\epsilon) \neq 1, \quad \forall \epsilon > 0. \)

Thus, \((T, S)\) is not semi-compatible.

**Remark 2.2.** In view of above example, it follows that the concept of semi-compatibility is more general than that of compatibility.

**Lemma 2.1.** [11] Let \(\{x_n\}\) be a sequence in a Menger space \((X, F, \ast)\) with continuous \(t\)-norm \(\ast\) and \(t \ast t \geq t\). If there exists a constant \(k \in (0, 1)\) such that

\[ F_{x_n x_{n+1}}(kt) \geq F_{x_{n+1} x_n}(t) \]

for all \(t > 0\) and \(n = 1, 2, 3, \ldots\), then \(\{x_n\}\) is a Cauchy sequence in \(X\).

3. **MAIN RESULT**

**Theorem 3.1.** Let \(A, B, S, T, L, M\) be self maps of a complete Menger space \((X, F, \ast)\) with \(t \ast t \geq t\) satisfying:

(3.1.1) \(L(X) \subseteq ST(X), M(X) \subseteq AB(X)\);

(3.1.2) \(AB = BA, ST = TS, LB = BL, MT = TM\);

(3.1.3) \(\text{either } L \text{ or } AB \text{ is continuous}\);

(3.1.4) \((L, AB) \text{ is semi-compatible and } (M, ST) \text{ weak compatible}\);

(3.1.5) \(\text{there exists a constant } k \in (0, 1) \text{ such that}\)

\[ F^2_{x_n x_{n+1}}(kt) \ast [F_{ABx_n x_{n+1}}(kt) \ast F_{STx_n x_{n+1}}(kt)] \geq [pF_{ABx_n x_{n+1}}(t) + qF_{ABx_n x_{n+1}}(t)] \cdot F_{ABx_n x_{n+1}}(2kt) \]

for all \(x, y \in X\) and \(t > 0\) where \(0 < p, q < 1\) such that \(p + q = 1\).

Then \(A, B, S, T, L, M\) have a unique common fixed point in \(X\).

**Proof.** Suppose \(x_0 \in X\). From condition (3.1.1) \(\exists\) \(x_1, x_2 \in X\) such that

\[ Lx_0 = STx_1 \quad \text{and} \quad Mx_1 = ABx_2. \]

Inductively, we can construct sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[ y_{2n} = Lx_{2n} = STx_{2n+1} \quad \text{and} \quad y_{2n+1} = Mx_{2n+1} = ABx_{2n+2} \text{ for } n = 0, 1, 2, \ldots. \]

**Step 1.** Taking \(x = x_{2n}\) and \(y = x_{2n+1}\) in (3.1.5), we have

\[ F^2_{2n+2n}(kt) \ast [F_{ABx_{2n} x_{2n+1}}(kt) \ast F_{STx_{2n+1} x_{2n+1}}(kt)] \geq [pF_{ABx_{2n} x_{2n+1}}(t) + qF_{ABx_{2n} x_{2n+1}}(t)] \cdot F_{ABx_{2n} x_{2n+1}}(2kt) \]

\[ F^2_{2n+2n} \ast [F_{2n+2n} \ast F_{2n+2n}] \geq [pF_{2n+2n} + qF_{2n+2n}] \cdot F_{2n+2n}(2kt) \]

\[ F_{2n+2n} \ast [F_{2n+2n} \ast F_{2n+2n}] \geq [pF_{2n+2n} + qF_{2n+2n}] \cdot F_{2n+2n}(2kt) \]

\[ F_{2n+2n} \ast [F_{2n+2n} \ast F_{2n+2n}] \geq [pF_{2n+2n} + qF_{2n+2n}] \cdot F_{2n+2n}(2kt) \]

Hence, we have

\[ F_{2n+2n}(kt) \geq F_{2n+2n}(t). \]

Similarly, we also have

\[ F_{2n+2n}(kt) \geq F_{2n+2n}(t). \]

In general, for all \(n\) even or odd, we have

\[ F_{n+2n}(kt) \geq F_{n+2n}(t) \]

for \(k \in (0, 1)\) and all \(t > 0\). Thus, by lemma 2.1, \(\{y_n\}\) is a Cauchy sequence in \(X\). Since \((X, F, \ast)\) is complete, it converges to a point \(z\) in \(X\). Also its subsequences converge as follows:

\[ \{Lx_{2n}\} \to z, \quad \{ABx_{2n}\} \to z, \quad \{Mx_{2n+1}\} \to z \text{ and } \{STx_{2n+1}\} \to z. \]
Case 1. Suppose AB is continuous.
As AB is continuous and \((L, AB)\) is semi-compatible, we get
\[ LABx_{2n+2} \rightarrow Lz \quad \text{and} \quad LABx_{2n+2} \rightarrow ABz. \] (3.1.7)
Since the limit in Menger space is unique, we get
\[ Lz = ABz. \] (3.1.8)

Step 2. By taking \( x = ABx_{2n} \) and \( y = x_{2n+1} \) in (3.1.5), we have
\[
F^2_{LABx_{2n}, Mx_{2n+1}}(kt) \ast [F_{ABx_{2n}, LABx_{2n}}(kt), F_{STx_{2n+1}, Mx_{2n+1}}(kt)] \\
\geq [pF_{ABx_{2n}, LABx_{2n}}(t) + qF_{STx_{2n+1}, Mx_{2n+1}}(2kt)].
\]
Taking limit \( n \rightarrow \infty \),
\[
F^2_{z, ABz}(kt) \ast [F_{ABz, ABz}(kt), F_{z, z}(kt)] \\
\geq [pF_{ABz, ABz}(t) + qF_{z, z}(2kt)].
\]
for \( k \in (0, 1) \) and all \( t > 0 \). Thus, we have
\[ z = ABz. \]

Step 3. By taking \( x = z \) and \( y = x_{2n+1} \) in (3.1.5), we have
\[
F^2_{z, Lz}(kt) \ast [F_{z, Lz}(kt), F_{z, z}(kt)] \\
\geq [pF_{z, Lz}(t) + qF_{z, z}(2kt)].
\]
Taking limit \( n \rightarrow \infty \),
\[
F^2_{z, Lz}(kt) \ast [F_{z, Lz}(kt), F_{z, z}(kt)] \\
\geq [p + qF_{z, Lz}(t)].
\]
for \( k \in (0, 1) \) and all \( t > 0 \). Thus, we have
\[ z = ABz. \]

Step 4. By taking \( x = Bz \) and \( y = x_{2n+1} \) in (3.1.5), we have
\[
F^2_{LBz, Mx_{2n+1}}(kt) \ast [F_{ABz, LABz}(kt), F_{STx_{2n+1}, Mx_{2n+1}}(kt)] \\
\geq [pF_{ABz, LABz}(t) + qF_{STx_{2n+1}, Mx_{2n+1}}(2kt)].
\]
Since \( AB = BA \) and \( BL = LB \), we have
\[ L(Bz) = B(Lz) = Bz \quad \text{and} \quad AB(Bz) = B(ABz) = Bz. \]
Taking limit \( n \rightarrow \infty \), we have
\[
F^2_{z, Bz}(kt) \ast [F_{z, Bz}(kt), F_{z, z}(kt)] \\
\geq [p + qF_{z, Bz}(2kt)].
\]
for \( k \in (0, 1) \) and all \( t > 0 \). Thus, we have
\[ z = ABz. \]
\[ F_{z,Bz}(kt) \geq \frac{p}{1 - q} = 1 \]
for \( k \in (0, 1) \) and all \( t > 0 \).
Thus, we have \( z = Bz \).
Since \( z = ABz \), we also have \( z = Az \).
Therefore, \( z = Az = Bz = Lz \).

**Step 5.** Since \( L(X) \subseteq ST(X) \) there exists \( v \in X \) such that \( z = Lz = STv \).
By taking \( x = x_{2n} \) and \( y = v \) in (3.1.5), we get
\[
F^2_{Lx_{2n}^{2n}z,Mz}(kt)[F_{ABx_{2n}^{2n}Lx_{2n}^{2n}(kt)}F_{STv,Mv}(kt)]
\geq [pF_{ABx_{2n}^{2n}Lx_{2n}^{2n}(kt)} + qF_{ABx_{2n}^{2n}STv(t)}]F_{ABx_{2n}^{2n}Mv}(2kt).
\]
Taking limit as \( n \to \infty \), we have
\[
F^2_{z,Mv}(kt)[F_{z,Mv}(kt)] \geq [pF_{z,Mv}(t) + qF_{z,Mv}(t)]F_{z,Mv}(2kt)
\geq F_{z,Mv}(t).
\]
Thus, we have \( z = Mv \) and so \( z = Mv = STv \).
Since \((M, ST)\) is weakly compatible, we have \( STMv = MSTv \).
Thus, \( STz = Mz \).

**Step 6.** By taking \( x = x_{2n}, y = z \) in (3.1.5) and using Step 5, we have
\[
F^2_{Lx_{2n}^{2n}z,Mz}(kt)[F_{ABx_{2n}^{2n}Lx_{2n}^{2n}(kt)}F_{STz,Mz}(kt)]
\geq [pF_{ABx_{2n}^{2n}Lx_{2n}^{2n}(kt)} + qF_{ABx_{2n}^{2n}STz(t)}]F_{ABx_{2n}^{2n}Mz}(2kt)
\]which implies that, as \( n \to \infty \)
\[
F^2_{z,Mz}(kt)[F_{z,Mz}(kt)] \geq [pF_{z,Mz}(t) + qF_{z,Mz}(t)]F_{z,Mz}(2kt)
\geq [p + qF_{z,Mz}(t)]F_{z,Mz}(kt)
\geq p + qF_{z,Mz}(kt)
\geq F_{z,Mz}(kt) \geq \frac{p}{1 - q} = 1.
\]
Thus, we have \( z = Mz \) and therefore \( z = Az = Bz = Lz = Mz = STz \).

**Step 7.** By taking \( x = x_{2n}, y = Tz \) in (3.1.5), we have
\[
F^2_{Lx_{2n}^{2n}Tz,Mz}(kt)[F_{ABx_{2n}^{2n}Lx_{2n}^{2n}(kt)}F_{STz,Mz}(kt)]
\geq [pF_{ABx_{2n}^{2n}Lx_{2n}^{2n}(kt)} + qF_{ABx_{2n}^{2n}STz(t)}]F_{ABx_{2n}^{2n}Mz}(2kt).
\]
Since \( MT = TM \) and \( ST = TS \), we have \( MTz = TMz = Tz \) and \( ST(Tz) = T(STz) = Tz \).
Letting \( n \to \infty \), we have
\[
F^2_{z,Tz}(kt)[F_{z,Tz}(kt)] \geq [pF_{z,Tz}(t) + qF_{z,Tz}(t)]F_{z,Tz}(2kt)
\geq p + qF_{z,Tz}(kt)
\geq F_{z,Tz}(kt) \geq \frac{p}{1 - q} = 1.
\]
Thus, we have \( z = Tz \). Since \( Tz = STz \), we also have \( z = Sz \)
Therefore, \( z = Az = Bz = Lz = Mz = Sz = Tz \), that is, \( z \) is the common fixed point of the six maps.

**Case II. L is continuous.**

Since \( L \) is continuous, \( LLx_{2n} \to Lz \) and \( L(AB)x_{2n} \to Lz \).

**Step 8.** By taking \( x = Lx_{2n}, y = x_{2n+1} \) in (3.1.5), we have

\[
F_{Lx_{2n},My_{2n+1}}(kt) \geq [pF_{Ax_{2n},Lx_{2n}}(t) + qF_{Sx_{2n+1},My_{2n+1}}(t)]F_{Ax_{2n},My_{2n+1}}(2kt)
\]

Letting \( n \to \infty \), we have

\[
F_{z,Lz}(kt) \geq [pF_{z,Lz}(t) + qF_{z,Lz}(t)]F_{z,Lz}(2kt)
\]

which implies that

\[
F_{z,Lz}(kt) = 1.
\]

Thus, we have \( z = Lz \) and using Steps 5-7, we have \( z = Lz = Mz = Sz = Tz \).

**Step 9.** Since \( L \) is continuous, \( LLx_{2n} \to Lz \) and \( LABx_{2n} \to ABz \).

Since \( (L, AB) \) is semi-compatible, \( L(AB)x_{2n} \to ABz \).

Since limit in Menger space is unique, so \( Lz = ABz \) and using Step 4, we also have \( z = Bz \).

Therefore, \( z = Az = Bz = Sz = Tz = Lz = Mz \), that is, \( z \) is the common fixed point of the six maps in this case also.

**Step 10.** For uniqueness, let \( w (w \neq z) \) be another common fixed point of \( A, B, S, T, L \) and \( M \).

Taking \( x = z, y = w \) in (3.1.5), we have

\[
F_{z,w}(kt) \geq [pF_{Ax,Lz}(t) + qF_{Sx,My}(t)]F_{Ax,My}(2kt)
\]

which implies that

\[
F_{z,w}(kt) \geq 1.
\]

Thus, we have \( z = w \).

This completes the proof of the theorem.

If we take \( B = T = I_X \) (the identity map on \( X \)) in theorem 3.1, we have the following:

**Corollary 3.2.** Let \( A, S, L \) and \( M \) be self maps of a complete Menger space \( (X, f, \ast) \) with \( t \ast t \geq t \) satisfying:

(a) \( L(X) \subseteq S(X), \ M(X) \subseteq A(X) \);
(b) \( L \) or \( A \) is continuous;
(c) \( (L, A) \) is semi-compatible and \( (M, S) \) is weak compatible;
(d) \( \exists k \in (0, 1) \) such that

\[
F_{Lx,Mx}(kt) \geq [pF_{Ax,Lx}(t) + qF_{Ax,Sy}(t)]F_{Ax,My}(2kt)
\]

for all \( x, y \in X \) and \( t > 0 \) where \( 0 < p, q < 1 \) such that \( p + q = 1 \).

Then \( A, S, L \) and \( M \) have a unique common fixed point in \( X \).
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