TWO METHODS OF ANALYSIS OF THIN ELASTIC OPEN HELICOIDAL SHELLS

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ABSTRACT
Thin open helicoidal shells are used in machine-building and in civil engineering. The numerical analysis of shell stress-strain state was fulfilled with the help of Runge-Kutta method. The analytical asymptotic method of analysis is presented for thin elastic open helicoidal shallow shells given in curvilinear non-orthogonal conjugate coordinates.

Keywords: open helicoid, shell, Runge-Kutta method, asymptotic method, stress-strain state, machine-building, civil engineering.

1. Introduction
Shells in the form of developable surfaces are the cheapest structures among thin-walled space structures of different geometrical forms due to their ability to be developed on a plane without any lap fold or break. The author made a careful study of all available scientific and technical books and papers on the geometrical investigations and on the stress-strain state analysis of developable shells (Krivoshapko, 1998) and helicoidal shells (Krivoshapko, 1999). It was found that only G.Ch. Bajoria (1985), S.B. Kositsyn (1996), M.K.K. Jayawardena (1992), and the author have studied stress-strain state of open helicoidal shells by analytic or numerical methods. But all thin-walled shells built in the form of open helicoids were designed on the basis of experimental data. In this paper, the author continues to study thin open helicoidal shells. Open helicoids are called as developable helicoids or Archimedes screw also.

2. Geometry of open helicoids
Assume a helix

\[ x = x(v) = a \cos v, \quad y = y(v) = a \sin v, \quad z = z(v) = bv \]

on the cylinder with the radius \( a \) as the cuspidal edge. Then the parametric equations of an open helicoid (Fig.1) can be written as

\[ x = x(u, v) = a \cos v - \frac{au \sin v}{m}, \]
\[ y = y(u, v) = a \sin v + \frac{au \cos v}{m}, \]
\[ z = z(u, v) = bv + \frac{bu}{m}, \quad m = \sqrt{a^2 + b^2}. \]

The arc length \( s \) of a helical cuspidal edge can be determined by the formula \( s = mv \) or \( v = s/m. \) Geometrical parameters \( a \) and \( b \) can be written in the form:

\[ a = a_0 \cos \phi, \quad b = a_0 \sin \phi \cos \phi. \]

Here \( a_0 \) is the radius of a plane development of the helix (1), \( \phi \) is the slanting angle of the rectilinear generators of the open helicoid (2). If we put these relations into the equations of an open helicoid (2), we can obtain the new parametrical equations of this surface.
\[ x = x(u, s) = a_0 \cos^2 \varphi (\cos \frac{s}{m} - \frac{u}{m} \sin \frac{s}{m}), \]
\[ y = y(u, s) = a_0 \cos^2 \varphi (\sin \frac{s}{m} + \frac{u}{m} \cos \frac{s}{m}), \]
\[ z = z(u, s) = (s + u) \sin \varphi, \quad m = a_0 \cos \varphi \]
or
\[ r = r(u, s) = x(u, s) \hat{i} + y(u, s) \hat{j} + z(u, s) \hat{k}, \]
where \( r \) is a radius-vector of a surface (3).

Using the equations of the surface (3) and the expressions
\[ ds^2 = dr^2 = A^2 du^2 + 2F du ds + B^2 ds^2 \]
and
\[ d^2 r \cdot n = L du^2 + 2M du ds + N ds^2, \]
where
\[ n = [\partial r / \partial u \times \partial r / \partial s]/[A^2 B^2 - F^2]^{1/2}, \]
one can obtain Gaussian quantities of the first and second orders in the theory of surfaces in the following form
\[ A = F = 1, \quad B^2 = 1 + \frac{u^2}{a_0^2} + \frac{u^2}{a_0^2}, \quad N = \frac{u \sin \varphi}{a_0^2 \cos \varphi}, \quad L = M = 0. \tag{4} \]

Let us rewrite all geometrical parameters necessary for design of the examined helicoidal surface:
\[ L = 2\pi \tan \varphi, \quad R_i = \sqrt{a_0^2 + u^2 \cos^2 \varphi}, \quad \tan \varphi_i = \tan \varphi / R_i = a_0 \sin \varphi \cos \varphi / R_i, \]
\[ s = \frac{va}{\cos \varphi} = va_0 \cos \varphi, \quad a_0 = \frac{a}{\cos^2 \varphi}, \quad a = \frac{m^2}{a_0^2}, \quad k = \frac{\cos^2 \varphi}{a_0^2}, \quad k_1 = k_\varphi = \tan \varphi, \quad k_2 = -\frac{1}{R_i}, \quad k_3 = \frac{N}{B^2} = \frac{u \tan \varphi}{a_0^2 B^2}, \quad \tan \chi = uk = \frac{u}{a_0}, \]
where \( L \) is the constant pitch of a helical directrix (1), \( R_i \) is a radius of a cylinder with a helix \( u = u_i = \text{const} \varphi_i \) is the slanting angle of the tangents of a helix \( u = u_i \) with the \( xOy \) plane, \( k \) is curvature of the cuspidal edge (1), \( k_1 \) and \( k_2 \) are the principal curvatures of the surface (3), \( k_3 \) is the curvature of the curvilinear coordinate \( s \); \( \chi \)
is an angle between two intersecting coordinate lines \( u \) and \( s \). The coordinate lines \( s = \text{const} \) are the rectilinear generatrices of an open helicoid.

It follows from equations (4) that the curvilinear coordinates \( u \) and \( s \) are non-orthogonal \( (F \neq 0) \) conjugate \( (M = 0) \) coordinates. In that case, Christoffel’s symbols in the theory of surfaces have the following form:
\[ \begin{bmatrix} 11 & 12 \\ 12 & 22 \end{bmatrix} = 0, \quad \begin{bmatrix} 12 & 22 \\ 12 & 22 \end{bmatrix} = -\begin{bmatrix} 12 \\ 22 \end{bmatrix} = \frac{1}{u} \begin{bmatrix} u \end{bmatrix}, \quad \begin{bmatrix} 22 \end{bmatrix} = -\frac{B^2}{u}. \]
The formulae for the determination of Christoffel’s symbols can be found in Goldenweizer’s monograph (1953). The formulae (4) satisfy to two equations of Peterson - Codazzi and to Gauss’s equation in the theory of surfaces. The values of quadratic forms (4) show that we must use the governing equations in curvilinear non-orthogonal conjugate coordinates.

3. GOVERNING EQUATIONS OF A SHELL THEORY

The system of governing equations suggested by Goldenweizer (1953) consists of six equations of equilibrium for a shell referred to any arbitrary coordinate system, six geometric equations, and eight physical equations. Let us write these twenty equations as applied to open helicoidal shells with middle surfaces (3). In our case we shall have 18 equations:

five equilibrium equations:
\[
\frac{\partial}{\partial u} (BN_u + S) + \frac{\partial}{\partial s} \left( \frac{N_s}{B} + S \right) - \frac{u}{a_0^2} \frac{N_s}{B} + \frac{u}{a_0} (X + \frac{Y}{B}) = 0,
\]
\[
\frac{1}{B} \frac{\partial}{\partial u} (BN_u + B^2 S) + \frac{\partial}{\partial s} \left( N_s + \frac{S}{B} \right) + \frac{u \tan \phi}{a_0^2 B} Q_s + \frac{u}{a_0} (Y + \frac{X}{B}) = 0,
\]
\[
- \frac{u \tan \phi}{a_0^2 B} N_s + \frac{\partial}{\partial u} (BQ_u) + \frac{\partial Q_s}{\partial s} + \frac{u}{a_0} Z = 0,
\]
\[
Q_s = \frac{a_0}{u} \left[ \frac{\partial}{\partial s} \left( M_s - \frac{M_{su}}{B} \right) - \frac{\partial}{\partial u} (M_u + BM_{us}) \right] + \frac{M_{su}}{a_0 B},
\]
\[
BQ_u = \frac{a_0}{u} \left[ B \frac{\partial}{\partial u} (BM_u) + \frac{\partial}{\partial u} (BM_{us}) + \frac{\partial}{\partial s} (BM_{su} - M_s) \right] - \frac{M_s}{a_0}, \tag{5}
\]
six geometric equations:
\[
\epsilon_u = \frac{\partial}{\partial u} (U_u + \frac{U_z}{B}), \quad \epsilon_s = \frac{1}{B} \frac{\partial}{\partial s} \left( U_s + \frac{U_z}{B} \right) + \frac{u}{a_0^2 B^2} \left( U_u + U_z \tan \phi \right),
\]
\[
\epsilon_{us} = \frac{u}{a_0 B^2} \left[ B^2 \frac{\partial}{\partial u} \left( \frac{U_z}{B} \right) + \frac{\partial U_u}{\partial s} - \frac{1}{u} U_u - \frac{\tan \phi}{u} U_z \right],
\]
\[
\kappa_u = \frac{\partial}{\partial u} \left[ \frac{a_0}{u} \left( \frac{\partial U_z}{\partial u} - \frac{1}{a_0 B^2} \frac{\partial U_{zu}}{\partial u} \right) + \frac{\tan \phi}{a_0 B^2} U_z \right] - \frac{1}{a_0 B^3} \left( \frac{\partial U_z}{\partial s} - \frac{u \tan \phi}{a_0 B^2} U_z \right),
\]
\[
\kappa_s = \frac{\partial}{\partial s} \left[ \frac{a_0}{u B} \left( \frac{\partial U_z}{\partial u} - \frac{1}{a_0 B^2} \frac{\partial U_{zu}}{\partial u} \right) - \frac{\tan \phi}{a_0 B^2} U_z \right] + \frac{1}{a_0 B} \frac{\partial U_z}{\partial u},
\]
\[
\kappa_{us} = \frac{\partial}{\partial s} \left[ \frac{a_0}{u} \left( \frac{\partial U_z}{\partial u} - \frac{\partial U_{zu}}{\partial u} \right) - \frac{u \tan \phi}{a_0 B^2} U_z \right] + \frac{1}{a_0 B^4} \left( \frac{\partial U_z}{\partial s} - \frac{u \tan \phi}{a_0 B^2} U_z \right),
\]
seven physical equations:
\[
N_{us} = C \frac{a_0 B}{u} \left( \epsilon_{us} - \frac{a_0}{u} \epsilon_{us} + \nu \epsilon_{su} \right),
\]
\[
M_{us} = -D \frac{a_0 B}{u} \left( \kappa_{us} + \nu \kappa_{su} \right),
\]
\[
S = \frac{Eh}{2(1 - \nu^2)} \left[ \frac{1 + B^2}{u^2} a_0^2 \epsilon_{us} - \frac{a_0}{u} (1 + \nu)(\epsilon_u + \epsilon_s) - \nu \epsilon_{us} \right],
\]
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\[ M_{us,stu} = \pm \frac{Eh^3}{12(1 + \nu)} \frac{a_0 B}{u} (\kappa_{us} - \kappa_{s,stu}), \]  

where \( D \) is flexural rigidity of the shell, \( C \) is the extensional rigidity of the shell, \( h \) is the shell thickness, \( \nu \) is Poisson’s ratio, \( E \) is Young’s modulus. The equations contain the normal \((N_{us})\) and the shear forces \((Q_{us})\), the bending \((M_{us})\) and the angular displacement \((\varepsilon_{us})\), parameters of curvature variations \((K_u, K_s)\), a parameter of torsion \((\kappa_{us})\), and three displacements \((U_u, U_s, U_z)\) which are the components of the elastic displacement vector \(U(u,s) = U_s r_A + U_s r_B - U_z n\). The vector of the external surface load \(P\) is decomposed in the form:

\[ P = X r_A + Y r_B - Z n. \]

The internal forces and moments containing in equations (5) are bound up with the internal forces \((N_u, N_s, S_u, S_s, Q_u, Q_s)\) and moments \((M_u, M_s, M_{us})\) obtained by decomposition of a vector of the internal forces and moments along the orthogonal unit vectors \((r_A, n, l = n \times r_A\) on the edge \(s = \text{const} \) and \(r_B, n, m = n \times r_B\) on the edge \(u = \text{const}\) by the following relations (Fig. 2):

\[
\begin{align*}
N_u \perp &= N_u \sin \chi, \quad N_s \perp = N_s \sin \chi, \\
S_u \perp &= S + N_u \cos \chi, \quad S_s \perp = S + N_s \cos \chi, \\
M_{us} \perp &= M_{us} \sin \chi, \quad M_{su} \perp = M_{su} \sin \chi, \\
M_u \perp &= M_u + M_{us} \cos \chi, \quad M_s \perp = M_s - M_{su} \cos \chi, \\
Q_u \perp &= Q_u, \quad Q_s \perp = Q_s.
\end{align*}
\]  

We have \(S_u \neq S_s\) and \(M_{us} \neq M_{su}\) because \(\chi \neq \pi / 2\). Using the last four physical equations (7) one can derive a new equation

\[ M_{us} + M_{su} \cos \chi (M_u - M_s) = 0 \]

or

\[ M_{us} + M_{su} + \tan^{-1} \chi (M_u - M_s) = 0. \]

4. **ONE-DIMENSIONAL PROBLEM FOR OPEN HELICOIDAL SHELLS**

Assume non-dimensional parameters

\[
\alpha = \frac{u}{a_0}, \quad U = \frac{U_u}{a_0}, \quad V = \frac{U_z}{a_0 B}, \\
W = \frac{U_z}{a_0}, \quad \mu = \tan \varphi, \quad t = \frac{h^2}{12 a_0^2}.
\]

Let us study the one-dimensional problem for an open helicoidal shell and that is why let us assume that all derivatives with respect to \(s\) are equal to zero. After putting the strains from the geometrical equations (6) into the eight physical equations (7), we may express the stress resultants in terms of the functions of displacements \(U, V,\) and \(W\). The expressions so obtained were substituted into the first three equations of equilibrium (5). The final result as the system of three ordinary eighth-order differential equations has the following form
\[
\frac{d}{d\alpha} \left[ \left( \frac{B^2}{\alpha} - \frac{1 + \nu}{2\alpha} \right) dU + \frac{1 - \nu}{2} \left( \frac{B^2}{\alpha} dV}{d\alpha} + 2V \right) + \nu \mu W \right] - \\
- \frac{1}{\alpha} \left( U + \mu W \right) + \frac{a_0 \alpha}{C} \left( X + \frac{Y}{B} \right) = 0,
\]

\[
\frac{d}{d\alpha} \left\{ \frac{1 - \nu}{2} \left( \frac{B^2}{\alpha} dU}{d\alpha} + \frac{B^4}{\alpha} dV}{d\alpha} - 2U - 2\mu W \right) + \\
+ \mu \left[ \frac{1}{\alpha} \left( \frac{B^2}{\alpha} dW}{d\alpha} + \mu V \right) + (1 - \nu) \mu \frac{d}{d\alpha} (\alpha V) \right] + \frac{a_0 \alpha B}{C} \left( Y + \frac{X}{B} \right) = 0,
\]

\[
\frac{d}{d\alpha} \left\{ \frac{t}{\alpha} \frac{d}{d\alpha} \left[ \frac{B^2}{\alpha} dW}{d\alpha} \right) + \mu \left( 2\alpha - \nu \alpha + \frac{1}{\alpha} \right) \frac{dV}{d\alpha} - \mu (1 + \nu V) \right] + \\
+ \mu \frac{1 - \nu}{2\alpha} \left( \frac{dU}{d\alpha} + B^2 \frac{dV}{d\alpha} \right) + \mu \frac{dU}{d\alpha} + \mu \nu (U + \mu W) \right) \right\} + \\
+ \frac{a_0 \alpha}{C} (\mu X + \frac{Y}{B} - Z) = 0.
\]

If one uses a shallow shell theory he must neglect the underlined terms of equations (10).

5. A NUMERICAL METHOD OF CALCULATION

Using the shallow shell theory, we can write the system of equations (10) in the form:

\[
\frac{d^2 U}{d\alpha^2} = -\frac{1}{\alpha} \frac{dU}{d\alpha} + \frac{U}{\alpha^2} + \mu \left( 1 - \frac{\nu}{B^2} \frac{1}{\alpha} \right) \frac{dW}{d\alpha} + \mu \frac{W}{\alpha^2} - \frac{a_0 \alpha^2}{CB^2} X,
\]

\[
\frac{d^2 V}{d\alpha^2} = \frac{1}{\alpha B^2} \left[ 2 \frac{dU}{d\alpha} - \frac{U}{\alpha} + \mu \left( 1 + \alpha^2 \frac{1 + \nu}{B^2} \right) \frac{dW}{d\alpha} - \mu \frac{W}{\alpha} \frac{(1 - 3\alpha^2)}{d\alpha} \right] - \\
- \frac{a_0 \alpha^2 [2BY + (1 + \nu)X]}{CB^4(1 - \nu)},
\]

\[
\frac{d^4 W}{d\alpha^4} = \frac{2}{B^2} \left( \frac{3}{\alpha} - \frac{1}{\alpha} \right) \frac{d^3 W}{d\alpha^3} + \frac{1}{B^4} \left( \alpha^2 - \frac{15}{\alpha^2} - 6 \right) \frac{d^2 W}{d\alpha^2} + \frac{1}{B^4} \left( \alpha + \frac{15}{\alpha^3} + 6 \frac{dW}{d\alpha} + \\
+ \mu \frac{\alpha^2}{tB^4} \left[ (1 - \nu) \frac{dV}{d\alpha} - \nu \frac{dU}{d\alpha} - U - \mu W \right] + \frac{a_0 \alpha^4}{tCB^4} Z.
\]

Assume a vector symbolism and rewrite the system of equations (11) in the form of a system of eight ordinary differential equations of the first order:

\[
y' = f(\alpha, y),
\]

(12)
\[
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5 \\
y_6 \\
y_7 \\
y_8 \\
\end{bmatrix} =
\begin{bmatrix}
U \\
U' \\
W \\
W' \\
V \\
V' \\
W'' \\
W''' \\
\end{bmatrix},
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6 \\
f_7 \\
f_8 \\
\end{bmatrix} =
\begin{bmatrix}
y_2 \\
y_2 \\
y_4 \\
y_5 \\
y_6 \\
y_7 \\
y_8 \\
y_8 \\
\end{bmatrix} =
\begin{bmatrix}
U' \\
U'' \\
W' \\
W'' \\
V' \\
V'' \\
W''' \\
W''' \\
\end{bmatrix}
\] (13)

Solving this system with the help of Runge–Kutta method one can determine the necessary numerical parameters. The numerical values of the stress resultants are computed with the help of the formulae:

\[
N_u = \frac{C}{\alpha} \left[ B y_2 + \frac{1}{B} \left( \frac{1}{\alpha} + v \alpha \right) (y_1 + \mu y_3) \right],
\]

\[
N_s = \frac{C B}{\alpha} \left[ \frac{y_1 + \mu y_3}{\alpha} - (1 - v) y_6 + v y_2 \right],
\]

\[
S = \frac{C}{2\alpha} \left[ (1 - v) B^2 y_6 - (1 + v) y_2 - \frac{2}{\alpha} (y_1 + \mu y_3) \right],
\]

\[
M_u = -\frac{D}{a_0 \alpha} \left[ \frac{B^2}{\alpha} y_7 + \left( v - \frac{1}{\alpha^2} \right) y_4 + \frac{\mu y_6}{\alpha} \right],
\]

\[
M_s = -\frac{D}{a_0 \alpha} \left[ \nu \frac{B^2}{\alpha} y_7 + \left( 1 - \frac{v}{\alpha^2} \right) y_4 + \frac{\mu y_6}{\alpha} \right],
\]

\[
M_{su} = (1 - v) \frac{DB}{a_0 \alpha} \left( \frac{1}{\alpha} y_7 - \frac{y_4}{\alpha^2} + \frac{\mu y_6}{\alpha} \right),
\]

\[
M_{us} = -(1 - v) \frac{\alpha D}{a_0 B} \left( \mu y_6 \right),
\]

\[
Q_s = \frac{D}{a_0^2 \alpha^2} \left[ \frac{B^2}{\alpha} y_8 + \left( 1 - \frac{3}{\alpha^2} \right) \left( y_7 - \frac{y_4}{\alpha} \right) + \mu \left( B^2 - v \alpha^2 \right) \frac{d^2 V}{d\alpha^2} + \mu \left( 2 \alpha (1 - v) - \frac{1}{\alpha} \right) y_6 \right],
\]

\[
Q_u = -B Q_s + \mu \frac{(1 - v) D}{a_0^2 B} \frac{d}{d\alpha} \left( \alpha^2 \frac{d V}{d\alpha} \right). \tag{14}
\]

In the numerical example, the theory of shallow shells was applied and therefore the underlined terms of equations (14) were taken equal to zero.

6. AN ANALYTICAL METHOD OF CALCULATION

If one wants to use an analytical asymptotic method he must rewrite the equation (10) in the following form:
\[
\frac{d}{d\alpha} \left[ \alpha^3 \frac{d}{d\alpha} \left( \frac{U}{\alpha} \right) \right] = -a_0 \alpha^4 \frac{dW}{d\alpha} + \mu \left[ W - \alpha B^2 \left( 1 + \nu \alpha^2 \right) \right],
\]
\[
\frac{B^4}{\alpha} \frac{dV}{d\alpha} = -B^2 \frac{dU}{d\alpha} + 2U + 2\mu W - \frac{2a_0}{(1 - \nu)C} \left[ \alpha (BY + X) d\alpha + A_1 \right],
\]
\[
\frac{d}{d\alpha} \left[ \frac{t}{\alpha} \frac{d}{d\alpha} \left( \frac{B^4}{\alpha} \frac{d}{d\alpha} \left( \frac{1}{d\alpha} \right) \right) \right] + \frac{1 - \nu}{2\alpha} \left( \frac{dU}{d\alpha} + B^2 \frac{dV}{d\alpha} \right) + \mu \alpha \frac{dU}{d\alpha} + \nu (U + \mu W) \bigg] + \frac{a_0 \alpha}{C} \left( \mu X + \frac{Y}{B} - Z \right) = 0,
\]
where \(A_1\) is an arbitrary constant of integration. If the parameter \(\mu = \tan \varphi < 1\)
is assumed to be small, one can examine open helicoids with the slope angle \(\varphi\) of the rectilinear generators under
45° using the asymptotic method of small parameter. The system of differential equations (15) is written in a form
convenient for the application of the method of small parameters. Assume \(\mu = \tan \varphi < 1\) for the small parameter and
let us use the expansion of unknown quantities \(U, V,\) and \(W\) into power series in terms of \(\mu\) as
\[
U = U(\alpha, \mu) = \sum_{k=0}^{\infty} U_k(\alpha) \mu^k,
\]
\[
V = V(\alpha, \mu) = \sum_{k=0}^{\infty} V_k(\alpha) \mu^k,
\]
\[
W = W(\alpha, \mu) = \sum_{k=0}^{\infty} W_k(\alpha) \mu^k,
\]
where \(U_k, V_k,\) and \(W_k\) are the vector coefficients subject to calculation.

Integrating the equations (15) we can obtain the general expressions for calculation of the vector coefficients
\(U_k, V_k,\) and \(W_k\)
\[
U = \mu a_0 \frac{Z}{C} \left[ \frac{\alpha^2}{3} + \frac{1}{2} - \frac{B^2}{2\alpha} \right] \arctan \alpha + \mu \alpha K_2 + \frac{A_2}{\alpha} + \alpha A_3,
\]
\[
V = \mu \left[ \left( \alpha^2 - 1 \right) \frac{K_2}{B^4} - \frac{K_1}{\alpha^2 B^2} + \frac{2\alpha}{B^4} \right] W d\alpha + \frac{1}{2C} \frac{1}{B^2 (1 - \nu)} - \frac{1}{3B^2} + \frac{\ln B^2}{1 - \nu} + \frac{\arctan \alpha}{\alpha} \right] Z - \frac{A_1}{2B^2} - \frac{A_2}{\alpha B^2} - \frac{\alpha}{B^2} A_3 + A_4,
\]
\[
tW = \frac{a_0 Z}{64C} \left[ \alpha^4 + \mu \left( \frac{(1 - \nu)}{32} \right) A_1 \left( \ln B^2 \right)^2 + \frac{1 - \nu}{4} \left( A_2 - A_3 \right) \arctan \alpha \left( \ln B^2 + \frac{7}{2} + \frac{\alpha^2}{2} \right) - \frac{7}{2} \right] - \frac{1}{2} \]
it is possible to obtain every time the three formulae for the determination of vector coefficients

$$
\begin{align*}
-\int \frac{\ln B^2}{B^2} \, d\alpha &+ \left(1 + \nu \right) \frac{A_1}{3} \left[\frac{\arctan \alpha}{4} \left(3\alpha^2 + 5\right) - \frac{5}{4} \alpha - \frac{\alpha^3}{3}\right] + \\
+ \mu^2 \frac{a_0 Z}{6C} \left\{(1 + \nu) \alpha \arctan \alpha \left(\frac{\alpha^2}{3} + \frac{5}{4}\right) - \frac{1}{8} \left(\arctan \alpha\right)^2 \left(5 + 3\alpha^2\right)\right\} - \\
(2\nu + 1) \frac{\alpha^4}{32} - \frac{5}{16} \left(\ln B^2\right)^2\right\} + \\
+ \mu^2 \alpha \int \left[\frac{\alpha^4}{B^4} \left(1 - \nu \right) - \frac{\alpha^2 K_2}{B^2} - \frac{1 + \nu \alpha^2}{B^2} \alpha W\right] \, d\alpha - \\
- \frac{\alpha^4}{B^4} K_2 \right\} \, d\alpha + C_1 B^2 \ln B^2 + C_2 \ln B^2 + C_3 \alpha^2 + C_4,
\end{align*}
$$

(17)

where

$$
E = \left[\frac{W - \alpha}{B^2} \left(1 + \nu \alpha^2\right) \frac{dW}{d\alpha}\right], \quad K_1 = \int E \, d\alpha, \quad K_2 = \int \frac{K_1}{\alpha^3} \, d\alpha,
$$

but $C_i$ and $A_i$ are eight arbitrary constants of integration. The expressions (17) were determined after taking the formulae into account.

$$
Z = \cos \varphi P_z, \quad X = -\sin \varphi P_z = -\mu Z, \quad Y = 0
$$

Substituting the power series (16) into equations (17) and considering step-by-step only terms with the same powers of $\mu$, it is possible to obtain every time the three formulae for the determination of vector coefficients $U_i$, $V_i$, and $W_i$. For example, taking $\mu = 0$ in formulae (17), we can find the first terms of series (16) in the form

$$
U_0 = \frac{A_{20}}{\alpha} + \alpha A_{30}, \quad V_0 = -\frac{A_{10}}{2B^2} - \frac{A_{20}}{\alpha B^2} - \frac{\alpha}{B^2} A_{30} + A_{40},
$$

$$
W_0 = a_0^3 Z \alpha^4 / (64D) + C_{10} B^2 \ln B^2 + C_{20} \ln B^2 + C_{30} \alpha^2 + C_{40},
$$

where $A_{i0} (i = 1 \div 4)$ and $C_{i0} (i = 1 \div 4)$ are the constants which can be found from the boundary conditions.

For the determination of the vector coefficients $U_i$, $V_i$, $W_i$ it is necessary after substituting the series (16) into the equations (17) to rewrite only the terms containing the $\mu$ parameter to the first power. So we shall have

$$
U_1 = \frac{a_0}{C} \left[\frac{\alpha^2}{3} + \frac{1}{2} - \frac{B^2 \arctan \alpha}{2\alpha}\right] + \alpha K_{2(0)} + \frac{A_{21}}{\alpha} + \alpha A_{31},
$$

$$
V_1 = \alpha \int \left[\left(\alpha^2 - 1\right) \frac{K_{2(0)}}{B^4} - \frac{K_{1(0)}}{\alpha^2 B^2} + \frac{2\alpha}{B^4} \frac{W_0}{W}\right] \, d\alpha +
$$

$$
+ \frac{a_0}{2C} \left[\frac{1}{1 - \nu} \left(\frac{1}{B^2} + \ln B^2\right) - \frac{1}{3B^2} + \frac{\arctan \alpha}{\alpha}\right] Z - \frac{A_{11}}{2B^2} - \frac{A_{21}}{\alpha B^2} - \frac{\alpha}{B^2} A_{31} + A_{41},
$$

$$
W_1 = \frac{1 - \nu}{8t} \left\{\frac{A_{20}}{4} \left(\ln B^2\right)^2 + \left(A_{20} - A_{30}\right) \left(2 \ln B^2 + 7 + \alpha^2\right) \arctan \alpha - \right.
$$

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\[-7\alpha - 2\left\{ \ln \frac{B^2}{B^2} \cdot d\alpha \right\} \right] + \frac{1 + \nu}{3t} A_{10} \left[ \frac{3\alpha^2 + 5}{4} \arctan\alpha - \frac{5}{4} \alpha^3 \right] +
\]
\[+ C_{11} B^2 \ln B^2 + C_{21} \ln B^2 + C_{31} \alpha^2 + C_{41},
\]
where \( A_{ij} (i = 1 \div 4) \) and \( C_{ij} (i = 1 \div 4) \) are the constants which can be found from the boundary conditions. The constants \( C_{ij} (i = 1 \div 4) \) are the known values because they were calculated earlier. By analogy one can continue calculating the next terms of the series (16). The internal forces and moments are determined with the help of formulae (14).

7. RESULTS
Two ways of static analysis of thin elastic long open helicoids are given for the application. The numerical method of calculation is based on the solution of the system of three ordinary eighth-order differential equations (11) with the help of Runge–Kutta method. All necessary formulae for the determination of the stress resultants are presented for shallow shells. For the application of asymptotic analytical method it is necessary to use the expansion of unknown parameters of displacements into the power series in terms of \( \mu \) (9). If two methods are available it is possible to control the results obtained. The both methods are easy to realize in computer programs.

8. DISCUSSION
The example of stress analysis of the developable helicoids presented by S.B. Kositsyn (1996) is fulfilled with the help of FEM. Bajoria (1985) and Jayawardena (1992) used equilibrium equations and physical equations containing internal forces \( N_u, N_s, S_x, S_y, Q_u, Q_s \) and moments \( M_u, M_s, M_{us}, M_{su} \) (Fig. 2). At present, the postgraduate students of the author work on a computer program which will give an opportunity to carry out static analysis of open helicoidal shells using the theoretical elaborations presented in this paper. The interesting numerical results of shell analysis will be passed into their paper.

9. CONCLUSIONS
Mechanical engineering is the main sphere of application of open helicoidal shells (Lewis J.R. and his colleagues, 1985; Kermis L.H., 1978; Zhao Y. and his colleagues, 2010). Methods presented in this paper can be used for static analysis of screw conveyors or ramps of many-storied garages.

R. Bradshaw with his colleagues (2002) wrote that at present, shells have lost their popularity compared to their heyday in the 1950s and 1960s. But there are signs, however, that shells are attracting interest among the new generation of architects and engineers and they will regain some of their former popularity when used appropriately.

REFERENCES

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