ESTIMATIONS ON VaR FOR OPTIONS: EXTENSION OF DELTA-GAMMA METHOD

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ABSTRACT
This paper provides an analytical formula for estimating the Value-at-Risk (VaR) of options. Based on the Black-Scholes option pricing formula, the increment of the option price is explicitly expressed in an infinite series by the Taylor expansion technique, and convenient full evaluations on VaR for options are obtained. Following this, the accuracy of the popularly used delta-gamma approximation is analytically and numerically examined by some examples. The results reveal that the delta-gamma method may underestimate or overestimate depending upon the ratio of the current asset and the strike price. Besides, in out-of-the-money cases, the examined results indicate by way of the delta-gamma method that while VaR misestimating is more significant for put options, they are less significant for call options.

Keywords: Black-Scholes formula, Delta-Gamma approximation, Option, Taylor expansion, Value-at-Risk.

1. INTRODUCTION
Modern financial theory is based on several important issues, one of which is focused on risk aversion. When the Basel Committee on Banking Supervision [1] announced in 1995 the decisions that banks may use internal models to estimate their market risks, a new challenge of risk management emerged. Financial institutions are facing the significant task of estimating and controlling their exposed market risks caused by asset prices changing. An algorithm integrating the overall risk level called Value-at-Risk (VaR), which combines all factors from different markets and different risks altogether into a single measurement, has recently become a popular standard for measuring market risks.

VaR was introduced to the industry in the late 80’s, in order to provide an estimation of the losses which can be realized by the end of a given time horizon with a given level of confidence. VaR could conveniently provide an actual amount which summarized the total risk in a portfolio of financial assets. Undoubtedly, RiskMetrics played an initiative role, and made VaR a popular risk management tool among practitioners, the academic community and especially among central banks.

For many years, derivative instruments and speculations became one of the major sources of hedging strategies. In order to measure the risk associated with complex derivatives, non-linear approaches on evaluating VaR may be more efficient. There are two main approaches used to calculate VaR in the non-linear case: One of them involves Monte Carlo simulation to obtain a numerical estimate of VaR; though this method is accurate but can be computationally expensive, in particular, for complicated derivatives. The second approach consists of analytical approximations of the value changes, providing an analytical formula and fast parametric solution to the problem. One of the most popular techniques to calculate VaR for a non-linear portfolio is the delta-gamma method. For detailed discussions and some implementations refer to Duffie and Pan [2] and Jorge and Andrew [3]. Essentially, the delta-gamma method provides a second order approximation of the value changes in the underlying risk factors. In order to obtain an accurate estimation, approximated errors are worthy of investigation. On the other hand, the accuracies of a general approach called Cornish-Fisher-approximations, which calculate up to the sixth moment, instead of only the first two moments, were investigated by Jaschke [4]. The results indicate that higher order may not always be more accurate. A lattice simulation procedure was proposed by Sorwar and Dowd [5] to evaluate financial risk for options. It avoids applying the delta–gamma and similar approximations.

Since the delta-gamma method provided calculation for VaR with an analytic formula which is easy to implement, further investigation on the accuracies will be discussed. For simplicity, in this study, attentions are focused on simple European options. Applying the Black-Scholes formula [6,7], the delta approximation, or the delta-gamma approximation is easy to compute. Intuitively, a better approximation may be achieved by incorporating more higher-order terms in the Taylor expansions. In this paper, full evaluation on VaR for European options is derived utilizing Taylor expansion up to infinite–orders rather than two-orders. The formula, essentially based on the Black-Scholes model, is explicitly presented, and the approximation errors arising from application of the delta-gamma method are analytically and numerically examined.

The remainder of this article is organized as follows. Based on the Black-Scholes formula, the VaR computation and complete Taylor expansions for the asset price change is presented in closed form. Numerical illustrations for
comparative analyses and the accuracy of the popular delta-gamma approximation are then investigated. In the final section some conclusions are discussed. Parts of the detailed proof of the analytic results are relegated to the Appendix.

2. THE MODEL BACKGROUND
Suppose the regular conditions for derivation of the Black-Scholes formula are satisfied, then the underlying asset price will follow a log-normal distribution, with mean, \( \ln S + (r - \sigma^2 / 2)T \) and standard deviation, \( \sigma \sqrt{T} \), denoted by,

\[
\ln S_T \sim \text{Normal} \left[ \ln S + \left( r - \frac{1}{2} \sigma^2 \right) T, \sigma \sqrt{T} \right].
\]

Here \( S \) denotes the underlying asset at the current time, \( S_T \) denotes the underlying asset at the future maturity date \( T \), \( K \) denotes some constant strike price, \( r \) denotes the riskless interest rate and \( \sigma \) denotes the volatility of the underlying stock return. After performing algebra, the current European call option price, which is the Black-Scholes formula, is expressed as:

\[
C = SN(d_1) - Ke^{-rT}N(d_2), \quad \text{with}
\]

\[
d_1 = \frac{\ln(S/k) + (r + \sigma^2/2)T}{\sigma \sqrt{T}}, \quad d_2 = d_1 - \sigma \sqrt{T}, \quad N(x) = \int_x^\infty \phi(u)du \quad \text{and}
\]

\[
\phi(u) = \exp\left(-u^2/2\right)/\sqrt{2\pi}.
\]

The Greek letters, delta and gamma for a call option are expressed as \( \Delta_C = N(d_1) \) and \( \Gamma_C = \phi(d_1)/\sigma S \sqrt{T} \), respectively. While the put option price is

\[
P = Ke^{-rT}N(-d_2) - SN(-d_1),
\]

and the corresponding delta and gamma Greek letters are

\[
\Delta_P = N(d_1) - 1 \quad \text{and} \quad \Gamma_P = \phi(d_1)/\sigma S \sqrt{T}.
\]

Thus, either for call option or put option, the gamma letter is essentially the same.

3. FULL EVALUATIONS ON VAR FOR EUROPEAN OPTIONS
3.1. VaR For Call Options
First, we evaluate the VaR for a European call option, with the current underlying asset price denoted by \( S_0 \). Since a lower asset price will cause the call option price to be reduced, the critical asset price occurs at the low side. Let \( S_* \) denote the \( \alpha \% \) critical asset price at the next day; that is the next day asset price, \( S_{*+1} \), will be higher than \( S_* \) with \( (1 - \alpha) \times 100\% \) confidence, say,

\[
\Pr(S_{*+1} > S_*) = 1 - \alpha.
\]

Applying Taylor expansion to formula (1) at \( S_0 \), the increment of the call option price results:

\[
\Delta C_* = \frac{\partial C}{\partial S} \times \Delta S_* + \frac{1}{2!} \frac{\partial^2 C}{\partial S^2} \times (\Delta S_*)^2 + \frac{1}{3!} \frac{\partial^3 C}{\partial S^3} \times (\Delta S_*)^3 + \ldots
\]

\[
= \Delta_C \times \Delta S_* + \frac{1}{2!} \Gamma_C \times (\Delta S_*)^2 + \sum_{m=1} \frac{1}{(m+2)!} \frac{\partial^{m+2} C}{\partial S^{m+2}} \times (\Delta S_*)^{m+2}.
\]

Here \( \Delta_C = \partial C/\partial S = N(d_1) > 0 \) and \( \Gamma_C = \partial^2 C/\partial S^2 = \phi(d_1)/\sigma S \sqrt{T} > 0 \). Hereafter, the right hand side of Taylor expansion is evaluated at \( S_0 \), the current asset price; for simplicity the notations are omitted. Moreover, \( \Delta S_* = S_{*+1} - S_0 \) where \( -\Delta S_* \) is the VaR for the underlying asset with \( (1 - \alpha) \times 100\% \) confidence. And \( \Delta C_* = C\big|_{S_*} - C\big|_{S_0} \), where \( -\Delta C_* \) is the VaR for the call option with \( (1 - \alpha) \times 100\% \) confidence.
Let $\gamma_m = \frac{\partial^m \Gamma_C}{\partial S^m}, a_m = (-1)^{m-1} \left( \lambda^2 \sum_{k=1}^{m} \frac{1}{k} - \beta \right)$, for $m \geq 1$, with $a_0 = \beta$, and $\lambda = 1 / \sigma \sqrt{T}$.

$\beta = 1 + d_1 / \sigma \sqrt{T} = 1 + \lambda d_1$, then

$$\gamma_{m+1} = \frac{\partial^m \Gamma_C}{\partial S^m} \left( \frac{\beta \Gamma_C}{S} \right) = \frac{\partial^m \Gamma_C}{\partial S^m} \left( \frac{\Gamma_C \times \beta}{S} \right)$$

$$= -m \sum_{k=0}^{m} C^m_k \times \frac{\partial \Gamma_C}{\partial S^k} \times \frac{\partial C^{m-k}}{\partial S^{m-k}} \left( \frac{\beta}{S} \right) = -m \sum_{k=0}^{m} C^m_k \times \gamma_k \times \frac{(m-k)!a_{m-k}}{S^{m-k+1}}$$

$$= -m! \left\{ \sum_{k=0}^{m} \gamma_k \times \frac{a_{m-k}}{k!S^{m-k+1}} \right\}.$$

For detailed derivations, please refer to the Appendix. Therefore, a recursive formula for $\{\gamma_i\}$ are obtained and expressed as:

$$\gamma_{m+1} = -m! \left\{ \sum_{k=0}^{m} \gamma_k \times \frac{a_{m-k}}{k!S^{m-k+1}} \right\}.$$

For computational convenience, an alternative representation of $\gamma_m$ is derived as follows: Define $\delta_0 = 1$, then $\gamma_0 = \Gamma_C = \delta_0 \Gamma_C$. Moreover, let $c(k) = k$, for $k \geq 1$, $c(0) = 1$, and $(-1)! = 1$, then the representation of $\gamma_k$ can be re-written as follow (Proof is given in Appendix):

**Lemma 1:** $\gamma_m = (m-1)! \times \delta_m \times \frac{\Gamma_C}{S^m}$, for all $m \geq 0$, where $\delta_m = \sum_{k=0}^{m-1} a_{m-1-k} \delta_k$ and $a_m = (-1)^{m-1} \left( \lambda^2 \sum_{k=1}^{m} \frac{1}{k} - \beta \right)$, for $m \geq 1$, with $\delta_0 = 1$ and $a_0 = \beta$.

By result of Lemma 1,

$$\frac{\partial^{m+2} C}{\partial S^{m+2}} = \frac{\partial^m C}{\partial S^m} \left( \frac{\partial^2 C}{\partial S^2} \right) = \frac{\partial^m \Gamma_C}{\partial S^m} = \gamma_m = (m-1)! \times \delta_m \times \frac{\Gamma_C}{S^m}.$$

Moreover, formula (3) could be re-written as:

$$\Delta C_* = \Delta C \times \Delta S_0 + \frac{1}{2!} \Gamma_C \times (\Delta S_0)^2 + \sum_{m=1}^{m} \frac{(m-1)!}{(m+2)!} \times \delta_m \times \frac{\Gamma_C}{S^m} \times (\Delta S_0)^{m+2}$$

$$= \Delta C \times \Delta S_0 + \frac{1}{2!} \Gamma_C \times (\Delta S_0)^2 + \Gamma_C \times (\Delta S_0)^2 \left\{ \sum_{m=1}^{m} \frac{\delta_m}{m(m+1)(m+2)} \times \left( \frac{\Delta S_0}{S} \right)^m \right\}.$$

Let $R_{c,n} = -\Gamma_C \times (\Delta S_0)^2 \left\{ \sum_{m=1}^{n} \frac{\delta_m}{m(m+1)(m+2)} \times \left( \frac{\Delta S_0}{S_0} \right)^m \right\}$ and $R_c = \lim_{n \to \infty} R_{c,n}$, then VaR for the call option is estimated by:

$$- \Delta C_* = -\Delta C \times \Delta S_0 - \frac{1}{2!} \Gamma_C \times (\Delta S_0)^2 + R_c.$$

Finally, full evaluation of VaR for the European call option is summarized as the following equation, which the asset price evaluated at $S_0$:

**Lemma 2:** $- \Delta C_* = -\Delta C \times \Delta S_0 - \frac{1}{2!} \Gamma_C \times (\Delta S_0)^2 + R_c$, where
\[ R_c = \lim_{n \to \infty} R_{c,n} \text{ and } R_{c,n} = -\Gamma_c \times (\Delta S_0)^2 \left\{ \sum_{m=1}^{n} \frac{\delta_m}{m(m+1)(m+2)} \right\} \left\{ \frac{\Delta S_n}{S_0} \right\}^m. \]

Also,

\[ \delta_m = -\sum_{k=0}^{m-1} \frac{a_{m-k} \delta_k}{c(k)}, \text{ for } m \geq 1, \text{ with } \delta_0 = 1; \text{ and } c(k) = k, \text{ for } k \geq 1, \text{ with } c(0) = 1; \]

\[ a_m = (-1)^{m-1} \left( \lambda^2 \sum_{k=1}^{m} \frac{1}{k} - \beta \right), \text{ for } m \geq 1, \text{ with } a_0 = \beta. \]

For practical usage, we may choose a suitable \( n \), and approximate \(-\Delta C_*\) by

\[ -\Delta C_* \approx \Delta \Gamma_c(n) = -\Delta_c \times \Delta S_0 - \frac{1}{2!} \Gamma_c \times (\Delta S_0)^2 + R_{c,n}. \]

This approach should be more accurate than the frequently used delta-gamma approximation, ignoring the remainder term, \( R_{c,n} \). Some numerical illustrations will demonstrate how to select a suitable \( n \) in the next section.

### 3.2. VaR For Put Options

Referring to VaR for put options, since higher asset prices will cause put option price to be decreased, therefore the critical asset price occurs at the high side. Let \( S^* \) denote the \( \alpha\% \) critical asset price at the next day; that is the asset price will be lower than \( S^* \) with \((1-\alpha)\times 100\%\) confidence, at the next day. Similarly, applying Taylor expansion to formula (2) at \( S_0 \), the increment of the put option price results:

\[ \Delta P^* = \frac{\partial P}{\partial S} \times \Delta S^* + \frac{1}{2!} \frac{\partial^2 P}{\partial S^2} \times (\Delta S^*)^2 + \frac{1}{3!} \frac{\partial^3 P}{\partial S^3} \times (\Delta S^*)^3 + \ldots \]

\[ = \frac{\partial P}{\partial S} \times \Delta S^* + \frac{1}{2!} \Gamma_p \times (\Delta S^*)^2 + \sum_{m=1}^{\infty} \frac{1}{(m+2)!} \frac{\partial^{m+2} P}{\partial S^{m+2}} \times (\Delta S^*)^{m+2}. \]

Here \( \Delta S^* = S_0 - S^* \), \( \Delta P = \partial P / \partial S = N(d_1) - 1 < 0 \) and \( \Gamma_p = \partial^2 P / \partial S^2 = \phi(d_1) / \alpha \sigma \sqrt{T} = \Gamma_c > 0 \).

Again, the right hand side of formula (4) is evaluated at \( S_0 \), the current asset price; \( \Delta P^* = P \big|_{S_0} - P \big|_{S^*} \), with \(-\Delta P^* \) is the VaR for put option with \((1-\alpha)\times 100\%\) confidence. It is worthy to note that since \( \Gamma_p = \Gamma_c \), thus \( \partial^k \Gamma_p / \partial S^k = \partial^k \Gamma_c / \partial S^k \), for any integer \( k \). An analog series expansion similar to Lemma 2, full evaluation of VaR for the European put option, is stated as follows:

**Lemma 3:** \(-\Delta P^* = -\Delta_p \times \Delta S^* - \frac{1}{2!} \Gamma_p \times (\Delta S^*)^2 + R_p \), where

\[ R_p = \lim_{n \to \infty} R_{p,n} \text{ and } R_{p,n} = -\Gamma_p \times (\Delta S^*)^2 \left\{ \sum_{m=1}^{n} \frac{\delta_m}{m(m+1)(m+2)} \right\} \left\{ \frac{\Delta S_n}{S} \right\}^m. \]

Here \( \{\delta_m\} \) are recursively obtained as stated in Lemma 2. Again, for practical usage, we may choose a suitable \( n \), and approximate \(-\Delta P^* \) by

\[ -\Delta P^* \approx \Delta \Gamma_p(n) = -\Delta_p \times \Delta S^* - \frac{1}{2!} \Gamma_p \times (\Delta S^*)^2 + R_{p,n}. \]

### 3.3. A Special Case: Near At-The-Money

In this subsection, VaR for call or put options in cases near at-the-money will be analytically discussed. As the underlying asset price gets close to the strike price, the value of \( \ln(S_0 / K) \) could be ignored, thus \( d_1 \) is
approximated by $(r / \sigma + \sigma / 2) \sqrt{T}$, and $\beta$ is approximated by $\tilde{\beta} = 1.5 + r / \sigma^2$, independent of the underlying asset price. And the representation of $a_m$ is reduced to $(-1)^m \tilde{\beta}$, then $\gamma_m$ is re-written as follows:

**Lemma 4:** It is true that $\tilde{\gamma}_m = (m - 1)! \tilde{\delta}_m \times \frac{\Gamma_m}{S^m}$, for all $m \geq 0$, where $\tilde{\delta}_m = -\tilde{\beta} \sum_{k=0}^{m-1} (-1)^{m-1-k} \tilde{\delta}_k / c(k)$, with $\tilde{\delta}_0 = 1$.

The derivation of Lemma 4 is similar to that of Lemma 1, thus proof is omitted. Then as both the underlying asset price and the strike price are close to each other, then the remaining terms of the Taylor expansion could be approximated by:

$$\tilde{R}_{c,n} = -\Gamma_c \times (\Delta S_c)^2 \left\{ \sum_{m=1}^{n} \frac{\tilde{\delta}_m}{m(m+1)(m+2)} \times \left( \frac{\Delta S_c}{S} \right)^m \right\}$$

for call option and

$$\tilde{R}_{p,n} = -\Gamma_p \times (\Delta S_p)^2 \left\{ \sum_{m=1}^{n} \frac{\tilde{\delta}_m}{m(m+1)(m+2)} \times \left( \frac{\Delta S_p}{S} \right)^m \right\}$$

for put options.

Moreover, to compare the magnitudes of $\tilde{R}_m$ under call option or put option, values of $\{\tilde{\delta}_m\}$ play an important role. Actually, $\tilde{\delta}_m$ depends on parameter $(r / \sigma^2)$ only, some further properties of $\tilde{\delta}_m$ are given as follows (Proof is given in Appendix):

**Lemma 5:** $\tilde{\delta}_{2k} > 0$ and $\tilde{\delta}_{2k+1} < 0$, for all $k \geq 0$.

For practical usage, $S_*$ will be less than the current asset price, $S_0$, thus $\Delta S_* < 0$. Combining results of Lemma 5 and the fact $\Delta S_* < 0$, then the following inequalities:

$$\tilde{\delta}_{2m} \times (\Delta S_*)^{2m} > 0$$

and

$$\tilde{\delta}_{2m+1} \times (\Delta S_*)^{2m+1} > 0,$$

are both obtained. Therefore, $\tilde{R}_{c,n} < 0$, is always true for $n > 0$; that implies the traditional delta-gamma approximation overestimates VaR for call options when the current asset price is close to the strike price.

On the other hand, $S^*$ will be greater than the current asset price, $S_0$, thus $\Delta S^* > 0$. Again, from results of Lemma 5 and the fact, $\Delta S^* > 0$, then the following inequalities resulted:

$$\tilde{\delta}_{2m+1} \times (\Delta S^*)^{2m+1} < 0$$

and

$$\tilde{\delta}_{2m} \times (\Delta S^*)^{2m} > 0.$$ 

Moreover, $\Gamma_c = \Gamma_p$, these results lead to the following comparisons about the misestimating of call options and put options.

**Lemma 6:** Suppose that the current asset price is close to the strike price, then $\tilde{R}_{c,n} < 0$ always holds. Moreover, suppose that $|\Delta S^*| = \Delta S_*$ is true, then $\tilde{R}_{c,n} < \tilde{R}_{p,n}$, for $n > 0$.

In brief, when the current underlying asset price is close to the strike price, the delta-gamma method overestimates VaR for call option; however, for the put option, the misestimating may be either overestimation or underestimation, the direction is not definite. Some numerical results examined in the next section will provide some partial evidence for Lemma 6.

4. **NUMERICAL ILLUSTRATIONS**

In this section, numerical comparisons between the delta-gamma approximation and the derived full evaluation on VaR for either European call options or put options are demonstrated: The strike price is set at $K = 100$, the yearly
volatility for returns of the underlying asset is set at, $\sigma = 30\%$, the yearly riskless interest rate is set at, $r = 1.5\%$, and the duration of the European option still has half a year, $T = 0.5$. For simplicity, suppose there are 250 trading days during a year, then the daily volatility is, $30% / \sqrt{250} = 1.897\%$. No matter if it is either the call option or the put option, the stated conditions remain the same. Some basic results for European options are presented in Table 1.

Table 1. Some Basic Results for European Options

<table>
<thead>
<tr>
<th>$S_0/K$</th>
<th>0.85</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
<th>1.05</th>
<th>1.10</th>
<th>1.18</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_c$</td>
<td>0.2661</td>
<td>0.3612</td>
<td>0.4600</td>
<td>0.5562</td>
<td>0.6448</td>
<td>0.7226</td>
<td>0.8216</td>
</tr>
<tr>
<td>$\Gamma_c$</td>
<td>0.0182</td>
<td>0.0196</td>
<td>0.0197</td>
<td>0.0186</td>
<td>0.0167</td>
<td>0.0144</td>
<td>0.0104</td>
</tr>
<tr>
<td>$\Delta_p$</td>
<td>-0.7339</td>
<td>-0.6388</td>
<td>-0.5400</td>
<td>-0.4438</td>
<td>-0.3552</td>
<td>-0.2774</td>
<td>-0.1784</td>
</tr>
<tr>
<td>$\Gamma_p$</td>
<td>0.0182</td>
<td>0.0196</td>
<td>0.0197</td>
<td>0.0186</td>
<td>0.0167</td>
<td>0.0144</td>
<td>0.0104</td>
</tr>
</tbody>
</table>

Note:
1. $K = 100$, yearly volatility $\sigma = 30\%$, $r = 1.5\%$, $T = 0.5$ year.
2. Call price $C = S_0 N(d_1) - Ke^{-rT} N(d_2)$; $\Delta_c = N(d_1); \Gamma_c = \phi(d_1)/\sigma S_0 \sqrt{T}$.
3. Put price $P = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$; $\Delta_p = N(d_1) - 1; \Gamma_p = \phi(d_1)/\sigma S_0 \sqrt{T}$.

For the call option, we temporarily set $\alpha = 5\%$, a 95% confidence level, the daily stock critical return rate is:

$$-1.645\sigma = -1.645 \times 30\% / \sqrt{250} = -3.1212\%;$$

then $S_0 = (1 - 3.1212\%) \times S_0$, where $S_0$ is the current asset price, and the magnitude of the critical change of asset price is, $\Delta S_* = S_* - S_0 = -3.1212\% \times S_0$. Then a proposed VaR estimate for the call option is estimated by:

$$\Delta \Gamma_c (n) = -\Delta_c \times \Delta S_* - \frac{1}{2!} \Gamma_c \times (\Delta S_*)^2 + R_{c,n},$$

where $R_{c,n} = -\Gamma_c \times (\Delta S_*)^2 \times \sum_{m=1}^{n} \frac{\delta_m}{m(m+1)(m+2)} \times \left( \frac{\Delta S_*}{S} \right)^m$, is evaluated at $S_0$.

In particular, as $n = 0$, $\Delta \Gamma_c (0) = -\Delta_c \times \Delta S_* - \frac{1}{2!} \Gamma_c \times (\Delta S_*)^2$, which is just the VaR estimate by the delta-gamma method. To evaluate the accuracy of the delta-gamma method in detail, some out-of-the-money, at-the-money, and in-the-money cases are examined, by setting ratios of the asset price to the strike price, from 0.85 to 1.18. That is the current asset price $S_0$ is assumed to range from 85 to 115. It is reasonable to expect that as the values of $n$ get sufficiently large, the proposed VaR estimates should be closed to each other. Actually, the numerical results indicate that as values of $n$ are beyond 3, VaR estimates almost converge. In order to be conservative, we regard $\Delta \Gamma_c (20)$ as the true VaR, that is $\lim_{n \to \infty} \Delta \Gamma_c (n)$. Then the accuracy of the delta-gamma method is evaluated by the following ratio:

Percentage Error$= 100\% \times \left[ \frac{\Delta \Gamma_c (20) - \Delta \Gamma_c (0)}{\Delta \Gamma_c (20)} \right]$. 

45
Table 2. VaR Estimations for European Call Options

<table>
<thead>
<tr>
<th></th>
<th>0.85</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
<th>1.05</th>
<th>1.10</th>
<th>1.18</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \Gamma_c(0)$</td>
<td>0.6419</td>
<td>0.9372</td>
<td>1.2774</td>
<td>1.6454</td>
<td>2.0235</td>
<td>2.3964</td>
<td>2.9554</td>
</tr>
<tr>
<td>$\Delta \Gamma_c(1)$</td>
<td>0.6432</td>
<td>0.9378</td>
<td>1.2770</td>
<td>1.6438</td>
<td>2.0209</td>
<td>2.3931</td>
<td>2.9519</td>
</tr>
<tr>
<td>$\Delta \Gamma_c(2)$</td>
<td>0.6433</td>
<td>0.9379</td>
<td>1.2771</td>
<td>1.6440</td>
<td>2.0210</td>
<td>2.3931</td>
<td>2.9514</td>
</tr>
<tr>
<td>$\Delta \Gamma_c(3)$</td>
<td>0.6433</td>
<td>0.9379</td>
<td>1.2771</td>
<td>1.6440</td>
<td>2.0210</td>
<td>2.3931</td>
<td>2.9514</td>
</tr>
<tr>
<td>$\Delta \Gamma_c(5)$</td>
<td>0.6433</td>
<td>0.9379</td>
<td>1.2771</td>
<td>1.6440</td>
<td>2.0210</td>
<td>2.3931</td>
<td>2.9514</td>
</tr>
<tr>
<td>$\Delta \Gamma_c(10)$</td>
<td>0.6433</td>
<td>0.9379</td>
<td>1.2771</td>
<td>1.6440</td>
<td>2.0210</td>
<td>2.3931</td>
<td>2.9514</td>
</tr>
<tr>
<td>$\Delta \Gamma_c(20)$</td>
<td>0.6433</td>
<td>0.9379</td>
<td>1.2771</td>
<td>1.6440</td>
<td>2.0210</td>
<td>2.3931</td>
<td>2.9514</td>
</tr>
<tr>
<td>Error(%)</td>
<td>0.2176</td>
<td>0.0730</td>
<td>-0.0250</td>
<td>-0.0872</td>
<td>-0.1224</td>
<td>-0.1377</td>
<td>-0.1352</td>
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$\alpha = 2.5\%$

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$\alpha = 0.5\%$

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Note:
1. $K = 100$, yearly volatility $\sigma = 30\%$, $r = 1.5\%$, $T = 0.5$ year; $\Delta \Gamma_c(0)$ is an estimation of VaR for call option under the delta-gamma method. In general, $\Delta \Gamma_c(n) = -\Delta_c \times \Delta S_c - \Gamma_c \times (\Delta S_c)^2 / 2 + R_{c,n}$, where

$$R_{c,n} = -\Gamma_c \times (\Delta S_c)^2 \times \sum_{n=1}^{n} \left[ \delta_n (m(m+1)(m+2)) \times (\Delta S_c)^m \right],$$

with $R_{c,0} = 0$.

2. Error=Percentage Error=100% $\times \left[ \Delta \Gamma_c(20) - \Delta \Gamma_c(0) \right] / \Delta \Gamma_c(20)$. 

Note:
1. $K = 100$, yearly volatility $\sigma = 30\%$, $r = 1.5\%$, $T = 0.5$ year; $\Delta \Gamma_c(0)$ is an estimation of VaR for call option under the delta-gamma method. In general, $\Delta \Gamma_c(n) = -\Delta_c \times \Delta S_c - \Gamma_c \times (\Delta S_c)^2 / 2 + R_{c,n}$, where

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with $R_{c,0} = 0$.

2. Error=Percentage Error=100% $\times \left[ \Delta \Gamma_c(20) - \Delta \Gamma_c(0) \right] / \Delta \Gamma_c(20)$. 

46
Table 3. VaR Estimations for European Put Options

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Error(%) $\alpha = 5\%$

|          | -0.0632 | -0.0235 | 0.0410 | 0.1309 | 0.2458 | 0.3848 | 0.6538 |

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Error(%) $\alpha = 2.5\%$

|          | -0.0887 | -0.0315 | 0.0613 | 0.1906 | 0.3560 | 0.5561 | 0.9436 |

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</table>

Error(%) $\alpha = 0.5\%$

|          | -0.1496 | -0.0477 | 0.1167 | 0.3455 | 0.6383 | 0.9930 | 1.6814 |

Note:
1. $K = 100$, yearly volatility $\sigma = 30\%$, $r = 1.5\%$, $T = 0.5$ year; $\Delta \Gamma_p(0)$ is an estimation of VaR for put option under the delta-gamma method. In general, $\Delta \Gamma_p(n) = -\Delta_p \times \Delta S^* - \Gamma_p \times (\Delta S^*)^2/2! + R_{p,n}$,

where $R_{p,n} = -\Gamma_p \times (\Delta S^*)^2 \times \sum_{m=1}^{n} \left[ \frac{\Delta S_i}{m(m+1)(m+2)} \right] \times \left( \frac{\Delta S^*}{S_0} \right)^n$, with $R_{c,0} = 0$.

2. Error=Percentage Error $= 100\% \times \left[ \frac{\Delta \Gamma_p(20) - \Delta \Gamma_p(0)}{\Delta \Gamma_p(20)} \right]$.

When the desired confidence levels $(1 - \alpha)100\%$ increase, values of VaR and percentage errors both increase. In the case of out-of-the-money, the delta-gamma method will underestimate the expected VaR. As ratios of the asset price to the strike price increases, the misestimating becomes slow-down; while as the ratio gets close to unity, the delta-gamma method becomes overestimate the expected VaR. Actually, the misestimating gets more serious as the asset price gradually over the strike price. Overall speaking, the misestimated percentage error is more serious in the out-of-the-money case. The discussed results with $\alpha$ set at 5\%, 2.5\% and 0.5\%, are exhibited in Table 2.

Referring to the put option with $\alpha = 5\%$, the daily stock critical return rate is
1.645\sigma = 1.645 \times 30\% / \sqrt{250} = 3.1212\%;
then \ S^* = (1 + 3.1212\%) S_0, \text{ where } S_0 \text{ are the current asset price, and the magnitude of the critical change of asset price is } \Delta S^* = S^* - S_0 = 3.1212\% \times S_0. \text{ Then a proposed VaR estimate for the put option is set as:}
\[ \Delta \Gamma_p(n) = -\Delta_p \times \Delta S^* - \frac{1}{2!} \Gamma_p \times (\Delta S^*)^2 + R_{p,n}, \]
where \ R_{p,n} = -\Gamma_p \times (\Delta S^*)^2 \sum_{m=1}^{n} \frac{\delta_m}{m(m+1)(m+2)} \left( \frac{\Delta S^*}{S} \right)^m, \text{ is evaluated at } S_0.
Similarly, we use \ \Delta \Gamma_p(20) \text{ as the true VaR}, \ \lim_{n \to \infty} \Delta \Gamma_p(n); \text{ then the accuracy of the delta-gamma method is evaluated by the ratio}
Percentage Error= 100\% \times \left[ \Delta \Gamma_p(20) - \Delta \Gamma_p(0) \right] / \Delta \Gamma_p(20).
No matter whether it is call option or put option, the percentage errors increase as the desired values of \alpha \text{ decrease, higher confidence needed. The same phenomena revealed as in the case of call option also occur for the put option:}
The delta-gamma method understimates the expected VaR in an out-of-the-money case, and overestimates the expected VaR in the case of in-the-money. However, for at-the-money case, the delta-gamma method overestimates the VaR for call option, but underestimates the VaR for put option. Furthermore, in case of at-the-money, the magnitudes of misestimating are serious for put options, with these results giving partial evidence to Lemma 6. It is worth noting that the misestimating of the delta-gamma approximation on VaR is rather significant in the case of out-of-the-money, in particular, for put options. The discussed results are shown in Table 3.

5. CONCLUSIONS
Following the financial tsunami experiences of 2008, the risk controls of derivative instruments on financial commodities have become tremendously important; in particular, more accurate estimations of downside potential for any portfolio are evidently very urgent. This paper provides an analytical formula, explicitly expressed for VaR estimates of European options. The essential idea is that by applying the Taylor expansion technique to the Black-Scholes option pricing formula, the increment of option price, with higher-order terms, instead of only two-order terms, is analytically derived.
The accuracy of the popular delta-gamma method is numerically examined by some examples. Overall, the results indicate that slight misestimating is discovered: No matter whether it is in cases of call options or put options, the delta-gamma method understimates the VaR in the out-of-the-money case and overestimates the VaR in the in-the-money case. While in the case of at-the-money, the delta-gamma method overestimates the VaR for call options, but underestimates the VaR for put options; in particular, the former phenomenon can be analytically proven. The percentage errors are more serious for the put options, compared to that of the call options. In order to get more accurate VaR estimates, in particular, in the case of out-of-the-money, we suggest that some extra terms in Taylor expansion in option price changes should be considered.

6. APPENDIX
Before developing the proofs, for convenience, some preliminary notations and properties are stated without proof as follows:
Property: (A1) \[ \frac{\partial N[h(x)]}{\partial x} = \phi[h(x)] \frac{\partial h(x)}{\partial x}, \quad N(x) = \int_{-\infty}^{x} \phi(u) du \quad \text{and} \quad \phi(u) = (2\pi)^{-1/2} \exp\left(-u^2 / 2\right). \]
(A2) \[ \frac{\partial d_1}{\partial S} = \frac{1}{\sigma S \sqrt{T}} = \frac{\lambda}{S}, \quad \text{where} \quad d_1 = \frac{\ln(S/K) + \left(r + \sigma^2 / 2\right) T}{\sigma \sqrt{T}} \quad \text{and} \quad \lambda = \left(\sigma \sqrt{T}\right)^{-1}. \]
(A3) \[ \frac{\partial^k \beta}{\partial S^k} = (-1)^{k+1} \frac{\lambda^2 (k-1)!}{S^k}, \quad \text{where} \quad \beta = 1 + \lambda d_1. \]
(A4) \[ \frac{\partial^k}{\partial S^k} \left( \frac{1}{S} \right) = (-1)^k \frac{k!}{S^{k+1}}. \]
Proof of Lemma 1:
\[ \frac{\partial^3 C}{\partial S^3} = \frac{\partial}{\partial S} \left( \frac{\lambda \phi(d_i)}{S} \right) = \lambda \phi(d_i) \times \frac{\partial}{\partial S} \left( \frac{1}{S} \right) + \frac{\lambda}{S} \times \frac{\partial}{\partial S} [\phi(d_i)] \]
\[ = -\frac{\lambda \phi(d_i)}{S^2} \left( \frac{\lambda}{S} \right)^2 [-d_i \phi(d_i)] = -\lambda \phi(d_i) \frac{\lambda}{S^2} (1+\lambda d_i) = -\beta \Gamma \frac{\lambda}{S} \]
\[ \frac{\partial^m \beta}{\partial S^m} \left( \frac{\beta}{S} \right) = \sum_{k=0}^{m} C_k^m \times \frac{\partial^k \beta}{\partial S^k} \times \frac{\partial^{m-k}}{\partial S^{m-k}} \left( \frac{1}{S} \right) \]
\[ = C_0^m \times \frac{\partial^m \beta}{\partial S^m} \left( \frac{1}{S} \right) + \sum_{k=1}^{m} C_k^m \times \frac{\partial^k \beta}{\partial S^k} \times \frac{\partial^{m-k}}{\partial S^{m-k}} \left( \frac{1}{S} \right) \]
\[ = \left( -1 \right)^m \frac{m! \beta}{S^{m+1}} + \sum_{k=1}^{m} C_k^m \left( -1 \right)^{k-1} \frac{\alpha^2 (k-1)!}{S^k} \left\{ \left( -1 \right)^{m-k} \frac{(m-k)!}{S^{m-k+1}} \right\} \]
\[ = \left( -1 \right)^m \frac{m! \beta}{S^{m+1}} + \left( -1 \right)^{m-1} \frac{\lambda^2 m!}{S^{m+1}} \times \sum_{k=1}^{m} \frac{1}{k} \]
\[ = \left( -1 \right)^{m-1} \frac{m! \beta}{S^{m+1}} + \lambda^2 \frac{m!}{S^{m+1}} \sum_{k=1}^{m} \frac{1}{k} - \beta = \frac{a_m}{S^{m+1}}, \]
here \( a_m = \left( -1 \right)^{m-1} \frac{m! \beta}{S^{m+1}} + \lambda^2 \frac{m!}{S^{m+1}} \sum_{k=1}^{m} \frac{1}{k} - \beta \), for \( m \geq 1 \), and \( a_0 = \beta \).

Let \( \gamma_m = \frac{\partial^m \Gamma_C}{\partial S^m} \), then
\[ \gamma_{m+1} = \frac{\partial^m \left( \frac{\partial \Gamma_C}{\partial S} \right)}{\partial S^m} = -\frac{\partial^m \left( \frac{\beta \Gamma_C}{S} \right)}{\partial S^m} = \frac{\partial^m \left( \Gamma_C \times \frac{\beta}{S} \right)}{\partial S^m} \]
\[ = -\sum_{k=0}^{m} C_k^m \times \frac{\partial^k \Gamma_C}{\partial S^k} \times \frac{\partial^{m-k}}{\partial S^{m-k}} \left( \frac{\beta}{S} \right) = \sum_{k=0}^{m} C_k^m \gamma_k \frac{a_{m-k}}{S^{m-k+1}}, \]
Define \( \delta_m = -\sum_{k=0}^{m-1} C_k^{m-1} a_{m-1-k} \delta_k \), with \( \delta_0 = 1 \); next, utilizing mathematical induction technique, the assertion of Lemma 1, that
\[ \gamma_m = \delta_m \times \frac{\Gamma_C}{S^m}, \text{ for all } m \geq 0, \]
will be proven.
As \( m=0 \), then \( \gamma_0 = \Gamma_C = \delta_0 \Gamma_C \); \( m=1 \), then \( \gamma_1 = \frac{\partial \Gamma_C}{\partial S} = -\frac{\beta \Gamma_C}{S} = \delta_1 \Gamma_C \), where \( \delta_1 = -\beta = -a_0 \);
\( m=2 \), then \( \gamma_2 = \frac{\partial^2 \Gamma_C}{\partial S^2} = -\sum_{k=0}^{1} C_k^1 \gamma_k \frac{a_{1-k}}{S^{1-k+1}} = -C_0^1 \gamma_0 \frac{a_1}{S^2} - C_1^1 \gamma_1 \frac{a_0}{S} \)
\[ = -C_0^1 \frac{a_1}{S^2} \Gamma_C - C_1^1 \delta_1 \frac{a_0}{S} = \delta_2 \frac{\Gamma_C}{S^2}, \text{ where } \delta_2 = -\left( C_0^1 a_1 + C_1^1 a_0 \delta_1 \right). \]
Suppose \( \gamma_k = \delta_k \frac{\Gamma_C}{S^k} \) is true, for all \( k \leq m \), then

49
\[ \gamma_{m+1} = \sum_{k=0}^{m} C_k^m \frac{a_{m-k}}{S_{m-k+1}} = -\sum_{k=0}^{m} C_k^m \left( \frac{\Gamma_C}{S^k} \right) \frac{a_{m-k}}{S_{m-k+1}} = -\frac{\Gamma_C}{S^{m+1}} \sum_{k=0}^{m} C_k^m a_{m-k} \delta_k = \delta_{m+1} \frac{\Gamma_C}{S^{m+1}}. \]

Therefore, the result of Lemma 1 is obtained.

**Proof of Lemma 5:**

The result of Lemma 5 will be proven by mathematical induction. By definition, \( \tilde{\delta}_0 = 1 > 0 \),
\[ \tilde{\delta}_1 = \tilde{\beta} = \left(3/2 + r/\sigma^2\right) < 0 \text{ and } \tilde{\delta}_2 = \tilde{\beta} \times \left\{ -\tilde{\delta}_0 + \tilde{\delta}_1 \right\} > 0. \]

Suppose that “\( \tilde{\delta}_{2k} > 0 \) and \( \tilde{\delta}_{2k+1} < 0 \)” are true for all \( k = 0, 1, 2, \ldots, m - 1 \); next, we prove that “\( \tilde{\delta}_{2m} > 0 \) and \( \tilde{\delta}_{2m+1} < 0 \)” are also correct. By definition \( \tilde{\delta}_{2m} \) and \( \tilde{\delta}_{2m+1} \) are expressed respectively as:
\[ \tilde{\delta}_{2m} = -\tilde{\beta} \sum_{k=0}^{2m-1} (-1)^{2m-1-k} \frac{\tilde{\delta}_k}{c(k)} \text{ and } \tilde{\delta}_{2m+1} = -\tilde{\beta} \sum_{k=0}^{2m} (-1)^{2m-k} \frac{\tilde{\delta}_k}{c(k)}. \]

Since \((-1)^{2m-1-k} \tilde{\delta}_k < 0\), for all \( 0 \leq k \leq 2m - 1 \), then \( \tilde{\delta}_{2m} > 0 \). On the other hand, \((-1)^{2m-k} \tilde{\delta}_k > 0\), for all \( 0 \leq k \leq 2m \), thus \( \tilde{\delta}_{2m+1} < 0 \). Therefore, results of Lemma 5 are obtained.

7. REFERENCES