BIOMETRIC SOLVENCY RISK FOR PORTFOLIOS OF GENERAL LIFE CONTRACTS (III) DEPENDENT LIVES

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ABSTRACT
We consider the endowment contract on two lives under mortality risk. In practice actuarial values of the tariff book are calculated under the simplifying assumption of independent future lifetimes. It is known that this assumption overestimates the joint-life net single and level premiums and underestimates the last-survivor net single and level premiums, where the maximal deviations are obtained by perfect positive dependence using the Höfdding-Fréchet upper bound. As a novel application, we discuss the impact of positive dependent lives on solvency calculations using a stochastic approach to the insurance risk and compare results with those of the current QIS5 standard approach. For a portfolio consisting of a single cohort of last-survivorship endowment contracts on two dependent lives, we show that the independence assumption underestimates insurance risk economic capital and calculate a non-parametric approximation of it based on the knowledge of the Spearman’s correlation coefficient as well as its maximal deviation from the independence assumption.

Keywords: Endowment insurance, joint-life, last-survivorship, Spearman’s coefficient, solvency capital

1. INTRODUCTION
Following the Solvency II project, there are two ways to calculate solvency capital requirements using the standard approach and internal models. Advanced insurance companies are likely to use internal models because they already have them in place. An internal model can improve consistency with respect to the obligations of the insurer and may lead to a lower (or higher) capital requirement than under the standard approach. Due to this, full acceptance of an internal model by the supervising authority must be made before the standard formula can be replaced by the internal model. It follows that the rationale for any internal solvency model must be rigorously funded and the made assumptions fully disclosed. Though the standard approach and an internal model have certainly many common features, there may be elements and risk components, which have not been accounted for in the standard approach. For a sound risk management it is necessary to develop models for full coverage of all risk components and understand any difference between the two approaches.

Within Solvency II “life insurance risk” is split into seven risk classes consisting of three biometric risk types (mortality risk, longevity risk and disability/morbidity risk), and four non-biometric risk types (lapse risk, expense risk, revision risk and catastrophe risk). The actuarial reserves of the life risks of biometric risk types are valued with biometric life tables (mortality and disability tables) while those of the non-biometric risk types require alternative valuation methods. The present work is restricted to the mortality/longevity risk type in a multiple life environment. The current standard requirements for the Solvency II life risk module have been specified in QIS5 [1], pp.147-163. QIS5 prescribes a solvency capital requirement (SCR), which only depends on the time of valuation (=time at which solvency is ascertained) but not on the portfolio size (=number of policies). It accounts explicitly for the uncertainty in both trends (=systematic risk) and parameters (=parameter risk) but not for the random fluctuations around frequency and severity of claims (=process risk). In fact, the process risk has been disregarded as not significant enough, and, in order to simplify the standard formula, it has been included in the systematic/parameter risk component. However, a precise mathematical modelling of the joint process and systematic risks for endowment contracts on two lives reveals that for small portfolio sizes and early times of valuation the QIS5 standard approach prescribes insufficient solvency ratios (see the examples in Section 7). For the purpose of internal models and improved risk management, it appears important to capture separately or simultaneously all risk components of biometric risks. A more detailed account of our contribution follows.

Section 3 recalls how the actuarial reserves of an endowment contract on two lives can be evaluated. In Section 4, the backward recursion formula for the actuarial reserves is derived. Then, using the sequence of random insurance losses for all future one-year term insurances with the sums at risk as death benefits and the famous Theorem of Hattendorff [2], the basic later required simple and convenient formulas for the mean and variance of the random insurance loss associated to an endowment contract on two lives are derived. Based on this we determine in Section 5 the conditional mean and variance of a portfolio’s prospective liability risk (=random present value of future cash-flows at a given time of valuation) and use a gamma distribution approximation to obtain the liability VaR and CVaR solvency capital as well as corresponding solvency capital ratios. These first formulas include only the
process risk and not the systematic risk. To include the latter risk in solvency risk investigation, we propose either to shift the biometric transition probabilities, as done in Section 6.2, or apply a stochastic model, which allows for random biometric transition probabilities, as explained in Section 6.3. Finally, Section 7 illustrates numerically and graphically the impact of the considered mortality/longevity VaR and CVaR risk capital models on the current Solvency II standard approach for endowment contracts on two lives.

2. A GENERAL PROSPECTIVE APPROACH TO THE LIABILITY RISK SOLVENCY CAPITAL

Starting point is a multi-period discrete time stochastic model of insurance. Given is a time horizon \( T \) and a probability space \( (\Omega, F, P) \) endowed with a filtration \( (F_t)_{t \geq 0} \) such that \( F_0 = \{ \Omega, \emptyset \} \) and \( F_T = F \) (think of \( F_t \) as the information available up to time \( t \)). Let \( L^c(F_t) \) be the space of essentially bounded random variables on \( (\Omega, F, P) \) and \( B^\infty \) the space of essentially bounded stochastic processes on \( (\Omega, F, P) \) which are adapted to the filtration \( (F_t)_{t \geq 0} \).

The basic discrete time stochastic processes are

\[ A_t, L_t : \text{the assets and actuarial liabilities at time } t \]

In a total balance sheet approach, their values depend upon the stochastic processes in \( B^\infty \), which describe the random cash-in and cash-out flows of any type of insurance business (non-life, life, pension insurance):

\[ P_{t-1} : \text{loaded premiums to be paid at time } t-1 \text{ (assumed invested at time } t-1) \]
\[ X_t : \text{insurance costs to be paid at time } t \text{ (includes insurance benefits, expenses} \]
\[ \text{and bonus payments paid during the time period } (t-1,t) \]
\[ R_t : \text{accumulation factor for return on investment for the time period } (t-1,t) \]

We assume that \( X_t \) is \( F_t \)-measurable and \( R_t \) is \( F_{t-1} \)-measurable. The random \( \text{cumulated accumulation factor} \)

for return over the period \( (s,t] \) \( 0 \leq s < t \leq T \), is denoted by \( R_{s,t} = \prod_{j=s+1}^{t} R_j \). Since \( R_t \) is \( F_{t-1} \)-measurable \( R_{s,t} \) is \( F_{t-1} \)-measurable, and therefore \( \{R_{s,t}, t > s\} \) is a predictable stochastic process. The quantity \( D_{s,t} = R_{s,t}^{-1} \) is called \( \text{random discount rate} \).

Consider \( F_{t-1} \)-measurable discrete time stochastic process \( \{CF_{j,t}, j = 0,1,...,T-t-1\} \) of future insurance cash-flows defined by

\[ CF_{j,t} = D_{s,t+1,t} \cdot X_{t+1} - P_{t+1} \quad j = 0,1,...,T-t-1. \quad (1) \]

The actuarial liabilities at time \( t \), also called time-\( t \) \( \text{prospective insurance liability} \), coincide with the random present value of all future insurance cash-flows at time \( t \) and are given by

\[ L_t = \sum_{j=0}^{T-t-1} D_{s,t+1,t} CF_{j,t}, \quad t = 0,1,...,T-1. \quad (2) \]

Using (1)-(2) and the relationship \( D_{s,t+1,t} = D_{s+1,t} \cdot D_{t+1,t+1} \), one shows the recursive equation

\[ L_{t+1} = (L_t + P_t) \cdot R_{t+1} - X_{t+1}, \quad t = 1,...,T-1. \]

On the other hand, the random assets over the time horizon \( [0,T] \) satisfy by definition the recursive equation \( A_{t+1} = (A_t + P_t) \cdot R_{t+1} - X_{t+1}, \quad t = 1,...,T-1. \) Through subtraction it follows that

\[ A_{t+1} - L_{t+1} = (A_t - L_t) \cdot R_{t+1}, \quad t = 1,...,T-1, \quad (3) \]

which implies the following equivalent probabilistic conditions (use that trivially \( L_T = 0 \))

\[ P(A_T \geq 0 | F_t) \geq 1 - \varepsilon, \quad (4) \]
\[ P(A_{t+1} \geq L_{t+1}, \tau = 1,2,...,T-t-1 | F_t) \geq 1 - \varepsilon, \quad (5) \]
\[ P(A_t \geq L_t | F_t) \geq 1 - \varepsilon. \quad (6) \]

Given a default probability \( \varepsilon > 0 \), the liability VaR solvency criterion (6) says that at time \( t \) the initial (deterministic) capital requirement \( A_t \) should exceed the random present value of future cash-flows with a probability of at least \( 1 - \varepsilon \). By (4)-(5) this criterion automatically implies that assets will exceed liabilities with the same probability at each future time over the time horizon \( T \). Let \( A_t^{\text{VaR}} = \text{VaR}_{1-\varepsilon} [L_t | F_T] \) be a minimum solution to (6), and assume that the best estimate insurance liabilities at time \( t \) coincide with the \text{net premium reserves} (in the sense defined later in (5.2)), that is let \( E[L_t | F_T] = V_t^Z \). Then, the liability VaR solvency capital
SC^{\text{VaR}}_t = A^{\text{VaR}}_t - V^Z_t = VaR_{t+\epsilon}[L_t|F_t] - V^Z_t \quad \text{represents the capital available at time } t \quad \text{to meet the insurance risk liabilities with high probability. A risk margin is added to this capital requirement (recall that in Solvency II the sum of the best estimate insurance liabilities and the risk margin determines the Technical Provisions). The liability VaR target capital is the sum of the liability VaR solvency capital and the risk margin defined by}

TC^{\text{VaR}}_t = SC^{\text{VaR}}_t + RM^{\text{VaR}}_t. \quad (7)

The cost-of-capital risk margin with cost-of-capital rate \( i_{CC} = 6\% \) is defined by

\[ RM^{\text{VaR}}_t = i_{CC} \cdot \sum_{k=1}^{T_t} v^k \cdot SC^{\text{VaR}}_{t+k}, \]

where \( T \) denotes the (maximum) time horizon, and \( v^f \) is the risk-free discount rate. For comparison with other solvency rules, one considers the value-at-risk solvency capital ratio at time \( t \) defined by

\[ SR^{\text{VaR}}_t = SC^{\text{VaR}}_t / V^Z_t. \]

Alternatively, let \( CVaR_{t+\epsilon}[L_t|F_t] = E[L_t|L_t > VaR_{t+\epsilon}[L_t|F_t]|F_t] \) be the conditional value-at-risk of the present value of future cash-flows at the confidence level \( 1 - e \) given the information available at time \( t \). The liability CVaR target capital \( TC^{\text{CVaR}}_t = CVaR_{t+\epsilon}[L_t|F_t] - V^Z_t + RM^{\text{CVaR}}_t = a_{RM}^{CVaR} \cdot V^Z_t + a_{CVaR}^{CVaR} \cdot CVaR_{t+\epsilon}[L_t|F_t] \) also meets the insurance risk liabilities and it defines the conditional value-at-risk solvency capital ratio at time \( t \) :

\[ SR^{\text{CVaR}}_t = SC^{\text{CVaR}}_t / V^Z_t. \quad (10) \]

3. CALCULATION OF ACTUARIAL RESERVES FOR THE ENDOWMENT ON TWO LIVES

The evaluation of the net actuarial reserves of a life insurance contract is based on biometric life tables. Denote by \( \omega \) the maximum attainable age of an insured life. Recall first the mortality table of a single life aged \( x \) and its probabilistic interpretation:

\( q_x \): probability a life aged \( x \) will die within one year (probability of death)
\( p_x = 1 - q_x \): probability a life aged \( x \) will survive to age \( x+1 \) (survival probability)
\( k \) \( p_x = \prod_{k=1}^{\omega-x} (1-q_{x+k-1}) \) \( q_{x+k-1} \): probability a life aged \( x \) will attain age \( x+k \) (survival probability)
\( k \) \( p_x = \prod_{k=1}^{\omega-x} (1-q_{x+k-1}) \) \( q_{x+k-1} \): probability a life aged \( x \) will die within the time period \( k = (k-1,k) \) \( k = 1, \ldots, \omega-x \)

The extension of life insurance for a single life to multiple lives is based on the notion of life status, for which there are definitions of survival and failure. To describe the endowment insurance on two lives aged \( x \) and \( y \) one requires probabilities for the joint-life status \((x:y)\) and the last-survivor status \((x:y)\). Let \( X, Y \) be the age-at-deaths of the lives \((x),(y)\) and \( T(x) = X = x, T(y) = Y = y \) the corresponding future lifetimes. Given the joint survival function \( S(x,y) = P(X > x, Y > y) \) of the couple \((X,Y)\), the survival probabilities of the future lifetimes \( T(x): y = \min(T(x),T(y)) \) and \( T(x): y = \max(T(x),T(y)) \) are obtained as follows:

\[ P_{x,y} = P(T(x): y > t) = P(X > x + t \land Y > y + t | X > x, Y > y) = \frac{S(x+t,y+t)}{S(x,y)} \quad (11) \]

\[ P_{x,y} = P(T(x): y > t) = P(X > x + t \lor Y > y + t | X > x, Y > y) = \frac{S(x+t,y) + S(x,y+t) - S(x+t,y+t)}{S(x,y)} \quad (12) \]

Though in general the random variables show a non-trivial dependence structure, it is common practice to assume for pricing purposes that the lives \((x)\) and \((y)\) are independent. In this simplified situation, the probabilities of survival depend on the life table of the single lives only and are denoted and given by

\[ P_{x,y} = P_{x+y}, P_{x} \cdot P_{y} \quad , P_{x+y} = P_{x+y} + P_{x+y} - P_{x} \cdot P_{y} \quad (13) \]
Remarks 3.1. Some time ago Youn et al. [3] have made a thorough analysis of the more general assumption of “partial independence” \( P_{x_1} \cdot P_{x_2} = P_{x_1} + P_{x_2} \) (suggested by Bowers et al. [4]), which simplifies much multiple life calculations. In particular, in case of a married couple, they show that this identity holds under the assumption that the mortality rate of the wife or the husband should not depend on whether they have a surviving spouse or not, nor on the surviving spouse’s age. This is generally assumed in practice. Insurance companies do not classify according to whether one has a surviving spouse or not, nor to spouse’s age. It can be shown that the partial independence assumption also holds for certain special survival functions. Among them one finds the “common shock survival model” defined by \( S(x, y) = S_1(x) \cdot S_2(x) \cdot R(\max(x, y)) \), where \( S_1(x), S_2(x), R(x) \) are survival distributions (see Denuit et al. [5] for an application), and the “Fréchet copula model” \( C(u, v) = (1 - \theta) \cdot u + \theta \cdot \min(u, v) \), \( \theta \in [0,1] \). Some further references on actuarial analysis of dependent lives include Carrière and Chan [6], Norberg [7], Youn and Shemyakin [8], Denuit and Cornet [9], Carrière [10], Dhaene et al. [11], Denuit et al. [12] and Hürlimann [13].

For a two life status \( u \) on lives aged \( x \) and \( y \) with random future lifetime \( T = T(u) \) one considers the curtate future lifetime discrete random variable \( K(u) = \left[ T(u) \right] \), which represents the number of completed future years lived by the status. For a time \( t > 0 \), let \( u + t \) denote the status obtained from \( u \) with lives aged \( x + t \) and \( y + t \). The random prospective loss at contract time \( t > 0 \) of a \( n \)-year endowment insurance on two lives with status \( u \) and sum insured \( SI \) is the random variable defined by

\[
L_t(u : n) = \left( v_{\min(K(u)+1:n)} - P(u : n) \cdot \frac{\bar{a}_n}{v_{\min(K(u)+1:n)}} \right) \cdot SI,
\]

where

\[
\bar{a}_n = \frac{1 - v^n}{i \cdot v}, v = \frac{1}{1+i}
\]

denotes a \( n \)-year annuity certain of one unit per year payable at the beginning of each year valued at the technical interest rate \( i \), and

\[
P(u : n) = \frac{1}{a(u : n)} - i \cdot v
\]

denotes the net (level) premium rates for a unit of sum insured, which are determined by the \( n \)-year life annuity net single premiums for the status \( u \) obtained from the formulas

\[
a(x : n) = \sum_{j=0}^{n-1} v^j \cdot p_x
\]

\[
a(x : y : n) = \sum_{j=0}^{n-1} v^j \cdot p_{xy}
\]

\[
a_x(x : y : n) = a(x : n) + a(y : n) - a(x : y : n)
\]

To define actuarial reserves properly, it is necessary to consider the possible states a status can take over future time. In the case of a single life aged \( x \) at contract time \( t = 0 \), one observes that with respect to the mortality risk the life can be in two different states \( \{1,2\} \) at time \( t > 0 \), which are defined by

\[
X_1 = 1 \iff (T(x) > t) \quad ((x) \text{ is alive at time } t > 0)
\]

\[
X_2 = 2 \iff (T(x) \leq t) \quad ((x) \text{ is dead at time } t > 0)
\]

Generalizing to a couple of lives aged \( x \) and \( y \) at \( t = 0 \), one observes that with respect to the mortality risk the couple can be in four different states \( \{1,2,3,4\} \) at time \( t > 0 \):

\[
X_1 = 1 \iff (T(x) > t, T(y) > t) \quad ((x) \text{ and } (y) \text{ are alive at time } t > 0)
\]

\[
X_2 = 2 \iff (T(x) > t, T(y) \leq t) \quad ((x) \text{ is alive } \text{ and } (y) \text{ is dead at time } t > 0)
\]

\[
X_3 = 3 \iff (T(x) \leq t, T(y) > t) \quad ((x) \text{ is dead } \text{ and } (y) \text{ is alive at time } t > 0)
\]

\[
X_4 = 4 \iff (T(x) \leq t, T(y) \leq t) \quad ((x) \text{ and } (y) \text{ are dead at time } t > 0)
\]
The mathematical reserve at time \( t > 0 \) of a \( n \)-year endowment insurance with a life status \((u)\) in state \( X_i = i \) at time \( t > 0 \) is defined to be the conditional expectation of the prospective loss given \((u)\) in state \( X_i = i \) at time \( t > 0 \), which is denoted and calculated as follows:

\[
V_i(u:n|x) = E[\mathcal{L}(u:n)|X_i = i], \quad i \in \{1,2,3,4\}. \tag{20}
\]

For a single life \((x)\) the mathematical reserve at the discrete time \( k \in \{0,\ldots,n-1\} \) in state \( X_k = 2 \) vanishes and the mathematical reserve in state \( X_k = 1 \) is given by (dropping as usual the index \( i = 1 \))

\[
V_1(x:n) = \left(1 - \frac{a(x+k:n-k)}{a(x:n)}\right) \cdot SI. \tag{21}
\]

Similarly, for a joint-life \((x:y)\) the mathematical reserves in the states \( X_k = 2,3,4 \) vanish and the mathematical reserve in state \( X_k = 1 \) is given by

\[
V_1(x:y:n) = \left(1 - \frac{a(x+k:y+k:n-k)}{a(x:y:n)}\right) \cdot SI. \tag{22}
\]

In contrast to this, for a last-survivor status \((x:y)\), only the mathematical reserve in the state \( X_k = 4 \) vanishes and the mathematical reserves in the states \( X_k = 1,2,3 \) are given by (similar to Bowers et al. [4], p. 501, for the case of continuous premium payments and death benefits payable at the time of death)

\[
V_k(x:y:n) = \left(1 - \frac{a(x+k:y+k:n-k)}{a(x:y:n)}\right) \cdot SI, \tag{23}
\]

\[
V_k(x:y:n|2) = \left(1 - \frac{a(x+k:n-k)}{a(x:y:n)}\right) \cdot SI, \tag{24}
\]

\[
V_k(x:y:n|3) = \left(1 - \frac{a(y+k:n-k)}{a(x:y:n)}\right) \cdot SI. \tag{25}
\]

Besides the mathematical reserves, which depend on the states of a status, one considers the net premium reserve at time \( t > 0 \), which is defined to be the conditional expectation of the prospective loss given survival to time \( t > 0 \) (similar Bowers et al. [4], Chap.17.7, p. 500):

\[
V_i(u:n) = E[\mathcal{L}(u:n)|\mathcal{T}(u) > t] = \sum_{i=1}^{n} V_i(u:n|x) \cdot P(X_i = i|\mathcal{T}(u) > t) \tag{26}
\]

For a single life \((x)\), respectively a joint-life \((x:y)\), the net premium reserve (26) coincides with the mathematical reserves (21) respectively (22). For a last-survivor status \((x:y)\), the net premium reserve is state independent and a probability weighted sum of the reserves (23) to (25) determined as follows (similar Bowers et al. [4], Chap.17.7, p. 502):

\[
V_k(x:y:n) = \sum_{i=1}^{4} p_i \cdot V_i(x:y:n|1) + p_i \cdot (1-p_i) \cdot V_i(x:y:n|2) + p_i \cdot (1-p_i) \cdot V_i(x:y:n|3). \tag{27}
\]

The obvious motivation for state-independent reserves is second-death life insurance, where during lifetime the insurer may not be informed about the first death. The concept of state independent reserve has been introduced by Frasier [14] (see also “The Actuary [15]” and Margus [16]) in the context of multiple life insurance for the last-survivor status. The choice between state independent and state dependent reserves depends upon loss recognition in the balance sheet (recognition or not of a status change). With state independent reserves, the insurance company administers the contract as if it had no knowledge of any decrements, as long as the contract is not terminated. Only the latter situation is considered in the present work (consult [17], Section 5, for further discussion).

4. THEOREM OF HATTENDORFF FOR THE ENDOWMENT ON TWO LIVES

The insurance loss random variable of the \( n \)-year endowment insurance on two lives with status \((u)\) and a unit of sum insured is given by formula (14) at the initial time \( t = 0 \):
\[ L(u : n) = v^{\min \{K(u) + 1, n\}} - P(u : n) \cdot \sum_{k=0}^{\min \{K(u), n\}} v^k. \]  

Now, using the indicator function and following Gerber et al. [18], Section 4, one writes

\[ v^{\min \{K(u) + 1, n\}} = \sum_{k=0}^{n} v^{k+1} I(K(u) = k), \quad \text{and} \quad \sum_{k=0}^{\min \{K(u), n\}} v^k = \sum_{k=0}^{n} v^{k} I(K(u) \geq k). \]

Hence, with the definition

\[ C_k(u : n) = v \cdot I(K(u) = k) - P(u : n) \cdot I(K(u) \geq k), \]

the equation (28) can be rewritten as

\[ L(u : n) = \sum_{k=0}^{n} v^k C_k(u : n). \]  

It is important to note that \( C_k(u : n) \) represents the random (net) cash-flow in contract year \( (k, k+1] \) valued at time \( k \). This shows that the insurance loss coincides with the random present value of all future cash-flows. Furthermore, for an arbitrary non-negative integer \( \tau = 0, 1, \ldots, n \), one defines the time-\( \tau \) prospective loss random variable

\[ L_\tau(u : n) = (1 + i) \cdot \sum_{k=\tau}^{n} v^k C_k(u : n), \]

whose (conditional) expected value defines the time-\( \tau \) actuarial reserve

\[ V_\tau(u : n) = E[L_\tau(u : n)|K(u) \geq \tau]. \]

It is clear that at the given discrete times (31) coincides with (14), (32) with (22) for the joint-life status \((u) = (x : y)\) and with (26) for the last-survivorship status \((u) = (x : y)\). In particular, one has \( L_0(u : n) = L(u : n) \) and \( V_0(u : n) = E[L(u : n)] \) is the initial actuarial reserve.

Let us now derive a recursion formula for the actuarial reserves. From the relationship

\[ v^\tau L_\tau(u : n) = v^\tau C_\tau(u : n) + \sum_{k=\tau+1}^{n} v^k C_k(u : n) = v^\tau C_\tau(u : n) + v^{\tau+1} L_{\tau+1}(u : n), \quad \tau = 0, 1, \ldots, n-1, \]

one gets the recursion formula for the random prospective loss

\[ L_\tau(u : n) = C_\tau(u : n) + vL_{\tau+1}(u : n). \]  

Inserting this into (32) yields

\[ V_\tau(u : n) = E[C_\tau(u : n)|K(u) \geq \tau] + vE[L_{\tau+1}(u : n)|K(u) \geq \tau]. \]  

Using (29) the first expectation in (34) can be rewritten as

\[ vE[I(K(u) = \tau)|K(u) \geq \tau] - P(u : n)E[I(K(u) \geq \tau)|K(u) \geq \tau] = vq_{u+\tau} - P(u : n), \]

with the usual notation \( q_{u+\tau} \) for the one-year probability of death of a two life status \((u + \tau)\). Conditioning on whether the status will survive another year, the second expectation equals

\[ E[L_{\tau+1}(u : n)|K(u) = \tau] \cdot \Pr(K(u) = \tau|K(u) \geq \tau) + \]

\[ E[L_{\tau+1}(u : n)|K(u) \geq \tau + 1] \cdot \Pr(K(u) \geq \tau + 1|K(u) \geq \tau) = 0 \cdot q_{u+\tau} + V_{\tau+1}(u : n) p_{u+\tau}, \]

with \( p_{u+\tau} = 1 - q_{u+\tau} \) the one-year survival probability. Inserting into (34), one obtains the backward reserve recursion formula (similar to Bowers et al. [19], equation (8.3.9))

\[ V_\tau(u : n) = vq_{u+\tau} - P(u : n) + vV_{\tau+1}(u : n) p_{u+\tau}. \]  

Rearranging yields the relationship

\[ P(u : n) = (1 - V_{\tau+1}(u : n)) q_{u+\tau} + (vV_{\tau+1}(u : n) - V_\tau(u : n)), \]

which decomposes the net premium into two components. The first one is the one-year term insurance premium payment for the net amount at risk \( \text{time-} (\tau + 1) \text{ sum at risk} \), also called time-\( \tau \) risk premium, and the second one is the time-\( \tau \) savings premium component that adjusts the actuarial reserve. Inserting (36) into (28) yields a generalized decomposition for the random insurance loss

\[ L(u : n) = v^{\min \{K(u) + 1, n\}} - \sum_{k=0}^{\min \{K(u), n\}} v^k \left[ (1 - V_{k+1}(u : n)) q_{u+k} + (vV_{k+1}(u : n) - V_k(u : n)) \right], \]

or equivalently
Now, following Gerber et al. [18], Section 7, define for an arbitrary integer \( k = 0, 1, \ldots, n - 1 \)
\[
\Lambda_k (u : n) = v (1 - V_{k+1} (u : n)) \left[ I (K(u) = k) - I (K(u) \geq k) \right] q_{u+k} .
\]  
(38)

If the status \( (u) \) is alive at time \( k \), (38) represents the random insurance loss for a one-year term insurance in the year \( k + 1 \) with death benefit equal to the time- \((k + 1)\) sum at risk (similar to Gerber [20], [21], Section 6.7, (7.2)). From this and (37) it follows that
\[
L(u : n) - V_0 (u : n) = \sum_{k=0}^{n-1} v^k \Lambda_k (u : n) .
\]  
(39)

A calculation using (38) yields the mean and variance formulas
\[
E[\Lambda_k (u : n)] = 0, \quad \text{Var}[\Lambda_k (u : n)] = v^2 (1 - V_{k+1} (u : n))^2 p_k q_{u+k} ,
\]  
(40)

with the usual probabilistic recursion \( p_{k+1} = p_k \cdot p_{u+k} \), \( p_0 = 1 \), \( k = 0, 1, \ldots \). On the other hand, the classical Theorem of Hattendorff [2], which is proved in Gerber et al. [18], tells us that the set of one-year insurance losses \( \{\Lambda_k (u : n)\} \) form a sequence of uncorrelated random variables
\[
\text{Cov}[\Lambda_j (u : n), \Lambda_k (u : n)] = 0, \quad 0 \leq j < k .
\]  
(41)

Together, this yields the following simple and convenient formulas for the mean and variance of the random insurance loss (28) associated to the endowment insurance on two lives
\[
E[L(u : n)] = V_0 (u : n), \quad \text{Var}[L(u : n)] = \sum_{k=0}^{n-1} v^{2(k+1)} (1 - V_{k+1} (u : n))^2 p_k q_{u+k} .
\]  
(42)

5. CALCULATION OF THE SOLVENCY II LIFE INSURANCE RISK ECONOMIC CAPITAL

We begin with risk calculations for a single endowment contract, and use them to determine the liability VaR and CVaR solvency capital for a portfolio of endowment contracts.

5.1. Target capital and solvency ratio for a single endowment contract

Given is a single endowment contract on two lives with random future cash-flows \( \{C_k (u : n)\} \) defined by (29). Risks not related to the lifetime of a contract are disregarded. In particular, expenses and related expense loadings are not taken into account. In the following, we assume that the status \( (u) \) is alive at contract time \( t \), which is the time of valuation for solvency assessment, that is \( K(u) \geq t \). First of all, one notes that the random present value of future cash-flows at time \( t \) defined by (note the simplification in notation)
\[
Z_t = \sum_{j=0}^{t-1} v^j C_{t+j} (u : n), \quad t = 0, 1, \ldots ,
\]  
(43)

coincides with the time- \( t \) prospective loss random variable defined in (31), that is \( Z_t = L_t (u : n), t = 0, 1, \ldots , n \).

In a first step, it appears useful to determine some moments of the conditional distribution of \( Z_t \) given \( K(u) \geq t \), say the conditional mean and variance. The variance formula (42) generalizes to an arbitrary discrete time of valuation \( t = 1, 2, \ldots, (\text{similar to Gerber [20], [21], Section 6.7, (7.11), Bowers et al. [19], Theorem 8.5.1.b}). Indeed, the derivation in Section 4 does not use any properties that are special to the date \( t = 0 \), except that the contract is alive at that time. Formula (39) generalizes as follows (similar to Bowers et al. [19], equation (8.5.9)):
\[
L_t (u : n) - V_0 (u : n) \cdot I (K(u) \geq t) = \sum_{k=t}^{n-1} v^k \Lambda_k (u : n) .
\]  
(44)

Noting further that \( Z_t = L_t (u : n), t = 0, 1, \ldots, n \), one obtains from (44) the following conditional mean and variance formulas
\[
E[Z_t | K(u) \geq t] = V_0 (u : n), \quad \text{Var}[Z_t | K(u) \geq t] = \sum_{k=t}^{n-1} v^{2(k+1)} (1 - V_{k+1} (u : n))^2 p_k q_{u+k} .
\]  
(45)

As shown in the next Subsection, these formulas can be used to determine the target capital and solvency ratio of a portfolio of endowment contracts using appropriate approximations for the distribution of the random present value of future cash-flows associated to this portfolio conditional on the contracts are still alive at the time of valuation.
5.2. Target capital and solvency ratio for a portfolio of endowment contracts
Towards the ultimate goal of solvency evaluation for an arbitrary life insurance portfolio, we consider now a portfolio of \( m \) endowment contracts on two lives. One observes that the \( i \)-th contract \( i \in \{1, \ldots, m\} \) is characterized by the following data set:

- ages \((x_i)\) and \((y_i)\) of the two insured lives at contract issue and status \((u_i)\)
- current status \((u_i + t_i)\) at the valuation time \( t \)
- sum insured of amount \( SI_i \)
- net premium \( NP(u_i : n_i) \cdot SI_i \) of the \( n_i \)-year endowment contract on two lives

To the \( i \)-th contract one associates (per unit of sum insured) its random future cash-flows \( \{C_k(u_i : n_i)\} \) as defined in (29), the corresponding \( L_{t_i} (u_i : n_i) \) time-\( t_i \) random prospective loss (31) and time-\( t_i \) actuarial reserve \( V_i (u_i : n_i) = E[L_{t_i} (u_i : n_i) | K(u_i) \geq t_i] \) obtained from (32). Then, the random present value of future cash-flows of the portfolio at valuation time \( t \) is obtained by multiplying (43) with the sums insured and summing over all contracts. It is given by

\[
Z_t = \sum_{i=1}^{m} \sum_{j=0}^{n_i-1} v^j C_{t+j} (u_i : n_i) SI_i, \quad t = 0, 1, \ldots, \max\{n_i - t_i\}.
\]

Similarly, summing the individual actuarial reserves, one gets the portfolio reserve

\[
V_t = \sum_{i=1}^{m} V_i (u_i : n_i) \cdot SI_i.
\]

Following Section 2, one defines the portfolio VaR solvency capital

\[
SC_{i V a R}^t = \text{VaR}_{t \geq \epsilon} \left[ Z_t | K(u_i) \geq t_i, i = 1, \ldots, m \right] - V_t,
\]

as well as the portfolio CVaR solvency capital

\[
SC_{i C V a R}^t = \text{CVaR}_{t \geq \epsilon} \left[ Z_t | K(u_i) \geq t_i, i = 1, \ldots, m \right] - V_t,
\]

and the corresponding solvency capital ratios

\[
SR_{i V a R}^t = \frac{SC_{i V a R}^t}{V_t}, \quad SR_{i C V a R}^t = \frac{SC_{i C V a R}^t}{V_t}.
\]

To determine these quantities it is necessary to determine the distribution of \( Z_t \) conditional on survival of the contract at time \( t \), and under the assumption that, conditional on any given mortality assumption, the remaining lifetimes of all contracts are independent of each other. First of all, using (45) one gets the mean and variance formulas

\[
E[Z_t | K(u_i) \geq t_i, i = 1, \ldots, m] = V_t,
\]

\[
\text{Var}[Z_t | K(u_i) \geq t_i, i = 1, \ldots, m] = \sum_{j=1}^{n_i-1} \sum_{k=0}^{\min(n_i-t_i-1,j)} v^{2(k+1)} \left[ \left( 1 - V_{t_i+k+1} (u_i : n_i) \right) \cdot SI_i \right]_{k+1} P_{u_i+t_i, q_{u_i+t_i+k}}.
\]

Though by Haitendorf’s Theorem in Section 4 the set of one-year insurance losses \( \{A_k (u_i : n_i)\} \) form a sequence of uncorrelated random variables for each contract \( i \in \{1, \ldots, m\} \), these random variables are not at all independent. This renders the evaluation of the distribution function of \( Z_t \) difficult, but not impossible. For example, one can use the De Pril [22] two stage recursive algorithm for this. As a more efficient and accurate solution, it is possible to approximate the latter using tight convex ordering bounds, as shown in [23]. It is known that the latter method approximates the CVaR solvency capital on the safe side because the CVaR risk measure preserves the convex risk ordering (e.g. [24], Theorem 1.1). Following a more pragmatic approach and for a wider use in the Solvency II project, we propose to approximate it in a first step using the mean and variance only. For sufficiently large life insurance portfolios, the use of an approximate gamma distribution turns out to be appropriate and can be justified (e.g. [25]). By abuse of notation, denote this approximation by \( F_i(x) = \Pr[Z_t \leq x | K(u_i) \geq t_i, i = 1, \ldots, m] \). Then, recalling the gamma distribution function, which is given by the incomplete gamma function, one has
\[ F_t(x) = G(\beta_t x; \alpha_t) = \frac{1}{\Gamma(\alpha_t)} \int_0^{\beta_t x} e^{-t} t^{\alpha_t - 1} dt, \quad \alpha_t = \frac{1}{k_t^2}, \quad \beta_t = \frac{1}{k_t^2 \mu_t}, \]

where the parameters \( \mu_t = V_t, k_t \) represent the conditional mean and coefficient of variation of \( Z_t \), and are obtained immediately from the formulas in (51). In this setting the solvency capital ratio formulas (50) take the analytical forms (5.23) in [25], (5.24) in [24], Section 4, or Furman and Landsman [26], Section 3)

\[
\begin{align*}
SR^\text{VaR}_t &= z_{1-\alpha} \left( k_t^{-2} \right) \cdot k_t^{-1} - 1, \\
SR^\text{CVaR}_t &= z_{1-\alpha} \left( k_t^{-2} \right) \cdot g \left( z_{1-\alpha} \left( k_t^{-2} \right) \right)
\end{align*}
\]

where \( z_{1-\alpha}(\alpha) := G^{-1}(1-\alpha) \) denotes the \((1-\alpha)\)-quantile of the standard normal distribution and \( g(x;\alpha) = G(x;\alpha) \) denotes its probability density.

The limiting results for a portfolio of infinitely growing size are similar to those in [27], Remark 5.1. If the coefficients of variation tend to zero, the gamma distributions converge to normal distributions and the solvency capital ratios converge to zero. This holds under the following assumption. Whenever insured status are independent and identically distributed and if the portfolio size is large enough then the ratio of observed status extinctions to portfolio size is close to the status extinction rate with high probability. This assumption defines the process risk as traditional insurance risk component in life insurance and describes the random fluctuations in the assumed status extinction probabilities derived from the survival probability model. However, in contrast to this, if the ratio of observed status extinctions to portfolio size is not close to the status extinction rate even for large portfolio sizes, systematic risk exists (e.g. Olivieri and Pitacco [28], Section 2.1). In this situation, which corresponds to the current Solvency II specification, the status extinction rate is uncertain and assumed to be random. This implies that in general stochastic models with "fluctuating basic probabilities", which are able to describe both the process and systematic risk components of life insurance risk, must be considered. This is the subject of Section 6.3.

6. COMPARING THE STANDARD APPROACH WITH VARIANTS OF THE STOCHASTIC APPROACH

The present Section has some overlap with [27], Section 6. It is therefore treated more briefly, but can be read independently. Recall that mortality risk in the current QIS5 standard approach accounts for the uncertainty in trends and parameters, the so-called systematic/parameter risk, but not for the process risk. Mortality risk applies to the class of insurance contracts contingent on death of the insured live with positive sums at risk, as typically encountered in term and whole life insurance products, and partially in endowment contracts. Similarly to the mortality risk, the longevity risk in QIS5 accounts only for the systematic/parameter risk component. Longevity risk applies to the class of insurance contracts contingent on survival of the insured live with negative sums at risk, as typically encountered in life annuity products and in the pure endowment component of endowment products.

In contrast to this, the portfolio solvency capital models of Section 5.2, when based on the traditional life table, only apply to the process risk. For full coverage of the mortality/longevity risks, that is coverage of the process and systematic/parameter risk components, these solvency models are revised and extended, where particular attention is put on a possible comparison with the current QIS5 standard approach. To take into account systematic/parameter risk in the models of Section 5.2, one can either shift the life table (see Section 6.2) or apply a stochastic mortality model, which allows for random mortality rates (see Section 6.3). For completeness the QIS5 standard approach is recalled in Section 6.1.

6.1. Solvency II standard approach

The current standard requirements for the Solvency II life risk module have been specified in QIS5 [1], pp.147-163. We begin with some general definitions and formulas, which apply to all life insurance risk types.

Given is a single life policy at time of valuation \( t \) with net premium reserve \( V_t \). Denote by \( V^\Delta_t \) the value of the reserves subject to a biometric shock \( \Delta \). The one-year solvency capital requirement (SCR) for this single policy is then given by

\[ SCR_t = V^\Delta_t - V_t. \]

Similarly to the decomposition (7) the Solvency II target capital (upper index S2 in quantities) is understood as the sum of the SCR and a risk margin defined by

\[ TC^{S2}_t = SCR_t + RM_t. \]
This formula includes a risk margin, which is calculated using a cost-of-capital approach with cost-of-capital rate \( i_{CoC} = 6\% \) as follows:

\[
RM_j = i_{CoC} \cdot \sum_{k=1}^{m} \left( V_{SCR_{jk}} \right),
\]

(57)

where \( T \) denotes the (maximum) time horizon, which may depend on the life policy, and \( v_j \) is the risk-free discount rate. Since Solvency II uses a total balance sheet approach, the defined single policy quantities must be aggregated on a portfolio and/or line of business level. For comparison with internal models it is useful to consider the solvency capital ratio at time \( t \) under the Solvency II standard approach defined by the quotient

\[
SR_{t}^{S2} = SCR_t / V_t^Z.
\]

(58)

Specializing to the mortality/longevity risk of endowment contracts on two lives with status \( (x) \) and \( (y) \), and the status \( (u) \), similarly to those introduced in Section 3. For example, the shifted life table of a single life aged \( (x) \) is defined by

\[
q_x^\Delta = (1 + \Delta) \cdot q_x - 1 - q_x^\Delta, \quad k^\Delta p_x = k^\Delta p_x \cdot \left( 1 - q_x^\Delta \right), \quad d^\Delta p_x = 1,
\]

(59)

with \( \Delta = 0.1 \) (10% increase in mortality rates at each age for the mortality risk) and \( \Delta = -0.25 \) (25% decrease in mortality rates at each age for the longevity risk). The shifted value \( V_t^\Delta \) of the portfolio reserve in (47) under a mortality/longevity shock is calculated with the formulas (22) and (26) using shifted survival probabilities for the evaluation of (16)-(19). One gets

\[
V_t^\Delta = \sum_{i=1}^{m} V_i^\Delta (u_i : n_i) \cdot SI_i.
\]

(60)

Finally, one gets from (55)-(58) and (60) the life insurance risk capital formulas

\[
SCR_t = V_t^\Delta - V_t, \quad TC_t^{S2} = SCR_t + RM_t, \quad RM_t = i_{CoC} \sum_{k=1}^{max(k< -1)} v_j^k \cdot SCR_{jk}, \quad SR_t^{S2} = \frac{TC_t^{S2}}{V_t}.
\]

(61)

6.2. Stochastic approach: shifting survival probabilities

Following the Sections 5.2 and 6.1, we consider the “shifted” random present value \( Z_t^\Delta \) of future cash-flows of the portfolio at time of valuation \( t \) with conditional mean and variance

\[
E[Z_t^\Delta \middle| K^\Delta(u_i) \geq t_i, i = 1, \ldots, m] = V_t^\Delta,
\]

\[
Var[Z_t^\Delta \middle| K^\Delta(u_i) \geq t_i, i = 1, \ldots, m] = \sum_{i=1}^{m} \sum_{k=0}^{m} v_i^{2(k+1)} \left[ (1 - V_t^\Delta) (u_i : n_i) \right] \cdot SI_i \cdot k^\Delta p_i \cdot q_i^\Delta \cdot q_i^\Delta,
\]

(62)

where \( K^\Delta(u) \) is the “shifted” curtate future lifetime random variable of a status \( (u) \) subject to the shifted survival probabilities. The distribution of \( Z_t^\Delta \) conditional on contract survival at time \( t \) is again approximated by a gamma distribution, which is denoted by

\[
F^\Delta(x) = \Pr [Z_t^\Delta \leq x \middle| K^\Delta(u) \geq t_i, i = 1, \ldots, m] = G(\beta^\Delta x; \alpha^\Delta), \quad \alpha^\Delta = \frac{1}{(k^\Delta)^2}, \quad \beta^\Delta = \frac{1}{(k^\Delta)^2} \cdot \mu^\Delta,
\]

(63)

where the conditional mean and coefficient of variation \( \mu^\Delta = V_t^\Delta, k^\Delta \) of \( Z_t^\Delta \) are obtained from the formulas (62). Making use of (53) and (54) one sees that the portfolio VaR and CVaR solvency capitals under the shifted survival probabilities are given by the expressions

\[
SCR_t^{\Delta, VaR} = VaR_{\phi(1-\varepsilon)} \left[ Z_t^\Delta \middle| K^\Delta(u) \geq t_i, i = 1, \ldots, m \right] - V_t = SCR_t + \left( z_{\Delta} \left( k^\Delta \right)^2 \right) \cdot \left[ V_t^\Delta \right] - V_t^\Delta,
\]

(64)

\[
SCR_t^{\Delta, CVaR} = CVaR_{\phi(1-\varepsilon)} \left[ Z_t^\Delta \middle| K^\Delta(u) \geq t_i, i = 1, \ldots, m \right] - V_t = SCR_t + \left( z_{\Delta} \left( k^\Delta \right)^2 \right) \cdot \left[ V_t^\Delta \right] - V_t^\Delta.
\]

(65)

The observations made in [27], Section 6.2, also hold in the present context. By small coefficients of variation the gamma distributions converge to normal distributions. This implies that the corresponding solvency capitals converge to those of normal distributions such that

\[
SCR_t^{\Delta, VaR} = SCR_t + \phi(1-\varepsilon) k^\Delta V_t^Z, \quad SCR_t^{\Delta, CVaR} = SCR_t + \frac{\phi(1-\varepsilon) k^\Delta V_t^Z}{\varepsilon}.
\]

(66)

Asymptotically, the solvency capital ratios tend to the following minimum values
By vanishing coefficients of variation the VaR and CVaR solvency capital ratios converge to the Solvency II solvency capital ratio. In this situation, the process risk has been fully diversified away, and, as expected, only the systematic/parameter risks remain.

6.3. Stochastic approach: Poisson-Gamma model for survival probabilities

In case the ratio of observed status extinctions to portfolio size is not close to the status extinction rate, even for large portfolio sizes, systematic risk exists. In this situation, the status extinction rate is uncertain and assumed to be random. Stochastic models with “fluctuating basic probabilities” are widespread in actuarial science and have been pioneered by Ammeter [29], [30]. In applications, the required bivariate survival probabilities in (1)-(2) are obtained from the copula approach, which uses the single life tables as marginal survival probabilities and a copula function to link them. Therefore, it suffices to restrict the attention to a single life table. Usually, one assumes a Bayesian Poisson-Gamma model such that the number of deaths is conditional Poisson distributed with a Gamma distributed random mortality which results in a negative binomial distribution for the unconditional distribution of the number of deaths. This model is well-known in non-life insurance, where it is used to describe the number of claims for a heterogeneous pool of risks in a static environment (e.g. Bühlmann [31]). It has also been used in [32], where the so-called “linear multivariate Poisson Gamma model” of risk theory has been introduced. Alternatively, one can assume that the annual number of deaths is conditional binomially distributed with a Beta distribution for the random mortality rate, which yields a negative hyper-geometric unconditional distribution, also called Binomial-Beta or Polya-Eggenberger model (e.g. Panjer and Willmot [33]). We note that the latter model has been adopted by Marocco and Pitacco [34] to describe the annual number of deaths in a portfolio of life annuities. From a mathematical point of view, the Poisson-Gamma choice has several advantages, in particular the generalization via compounding to multiple cohorts with varying amounts of benefits (e.g. [32]).

For the purpose of modelling simultaneously the process and the systematic risks, we consider the Poisson-Gamma model with time-dependence of the type introduced in Olivieri and Pitacco [28]. This probability model is able to up-date its parameters to experience, which should be quite attractive for the design of dynamic information based internal risk management systems. Given a fixed time of valuation \( t \in \{0, 1, \ldots, \nu - x\} \) and a mortality table for a life aged \( x \), which is based on an initial cohort of size \( \ell_{xst} \) at time \( t \). Let \( D_{xst-1} \) denote the random number of deaths produced by the cohort in the time period \( [t + \tau - 1, t + \tau], \tau = 1, 2, \ldots \). For the first time period \( \tau = 1 \), we assume that there is no experience available and that the random number of deaths is conditional Poisson distributed such that

\[
D_{xst} \sim \text{Po}(\ell_{xst}Q_{xst}), \quad Q_{xst} \sim \text{Gamma}\left(\alpha, \beta, \frac{q_{xst}}{\ell_{xst}}\right).
\]

It follows that the unconditional distribution of the number of deaths in the first time period is negative binomially distributed such that

\[
D_{xst} \sim \text{NB}\left(\alpha, \beta, \frac{q_{xst}}{\ell_{xst}}\right), \quad \ell_{xst} = \frac{\beta}{\alpha}, \quad Q_{xst} = \frac{\beta}{\alpha}.
\]

In contrast to the expected number of deaths \( \ell_{xst}q_{xst} \) predicted by the life table, one has

\[
E[D_{xst}] = \frac{\alpha}{\beta} \ell_{xst}q_{xst}.
\]

To model a systematic deviation from the life table expectation, as encountered in mortality/longevity risk assessments, one assumes that the quotient \( \alpha / \beta \) is different from one: greater than one for mortality risk and less than one for longevity risk. Suppose that at time \( t + 1 \), the number of deaths \( d_{xst} \) observed in the cohort over the first time period is available, and let \( \ell_{xst+1} = \ell_{xst} - d_{xst} \) be the up-dated cohort size. A calculation shows that the posterior distribution of \( Q_{xst} \) conditional on the information \( D_{xst} = d_{xst} \) is Gamma distributed

\[
Q_{xst}|D_{xst} \sim \text{Gamma}\left(\alpha + d_{xst}, \beta + \ell_{xst}q_{xst}\right),
\]

which reveals that the initial structural systematic risk parameters are up-dated as follows:
\[(\alpha, \beta) \rightarrow (\alpha + d_{xst}, \beta + \ell_{xst}q_{xst}).\]

Passing to the second time period \((t + 1, t + 2]\), we assume similarly to the first period that

\[D_{xst+1}|d_{xst} \sim \text{Po}(\ell_{xst}Q_{xst}), \quad Q_{xst+1}|d_{xst} \sim \text{Gamma}\left(\alpha + d_{xst}, \frac{\beta + \ell_{xst}q_{xst}}{q_{xst+1}}\right).\]

This implies an unconditional negative binomial distribution of the type

\[D_{xst+1}|d_{xst}, d_{xst+1}, \ldots, d_{xst-2} \sim \text{NB}\left(\alpha + \sum_{k=0}^{t-1} d_{xst-k}, \frac{\theta_1}{\theta_2 + 1}\right), \quad \theta_2 = \frac{\beta + \ell_{xst}q_{xst}}{\ell_{xst}q_{xst+1}}.\]

Iterating the above Bayesian scheme, one generalizes as follows. At time \(t + \tau - 1, \tau \geq 2\), having observed the annual number of deaths \(d_{xst}, d_{xst+1}, \ldots, d_{xst-2}\), the updated cohort size for the next time period \((t + \tau - 1, t + \tau]\) is obtained from the recursion \(\ell_{xst-k} = \ell_{xst-k-2} - d_{xst-k-2}, \ k = 2, 3, \ldots, \tau\). The corresponding unconditional negative binomial distribution of the number of deaths is then given by

\[D_{xst+\tau}|d_{xst}, d_{xst+1}, \ldots, d_{xst-2} \sim \text{NB}\left(\alpha + \sum_{k=0}^{t-1} d_{xst-k}, \frac{\theta_1}{\theta_2 + 1}\right), \quad \theta_2 = \frac{\beta + \sum_{k=0}^{\tau-1} \ell_{xst-k}q_{xst-k}}{\ell_{xst-k}q_{xst-k+1}}.\]

It is interesting to look at the expected number of deaths given the available information, which is given by and compares with the expected number from the life table as follows:

\[E[D_{xst+\tau}|d_{xst}, d_{xst+1}, \ldots, d_{xst-2}] = \frac{\alpha + \sum_{k=0}^{t-1} d_{xst-k}}{\beta + \sum_{k=0}^{\tau-1} \ell_{xst-k}q_{xst-k}} \ell_{xst-k}q_{xst-k} \leq E[D_{xst+\tau}] = \ell_{xst-1}P_{xst}q_{xst-\tau}.\]

If mortality experience is consistent with what is expected, the quotient of both expected values remains constant over time. On the other hand, if experience is better (worse) than expected, the same quotient will increase (decrease) over time. Clearly, such a systematic deviation must have an impact on the capital requirement for mortality (longevity) risk.

In practice one proceeds as follows. Given a fixed time of valuation \(t \in [0, 1, \ldots, \omega - x]\), consider the Poisson-Gamma probabilities of death obtained from (70) and (76) defined by

\[q_{xst}^{\text{PG}} = \frac{\alpha}{\beta}, \quad q_{xst}^{\text{PG}} = \frac{\alpha + \sum_{k=0}^{t-1} d_{xst-k}}{\beta + \sum_{k=0}^{\tau-1} \ell_{xst-k}q_{xst-k}}, \quad \tau = 1, 2, \ldots,\]

\[p_{xst}^{\text{PG}} = 1 - q_{xst}^{\text{PG}}, \quad p_{xst}^{\text{PG}} = 1 - q_{xst}^{\text{PG}} - \ell_{xst-k}q_{xst-k-1}, \quad k = 1, 2, \ldots, \tau = 1, 2, \ldots,\]

Replacing everywhere in the formulas (62)-(66) the superscript \(\Delta\) by the superscript \(\text{PG}\) and using formulas similar to (77) for both lives in calculations, as well as an appropriate copula linking the single life survival probabilities, one obtains portfolio VaR and CVaR solvency capital formulas under the Poisson-Gamma life tables similar to (64) and (65). Similar limiting results apply.

7. IMPACT OF INDEPENDENCE ASSUMPTION ON LIFE INSURANCE RISK

Since actuarial values of the tariff book have been calculated under the simplifying assumption of independent future lifetimes \(T(x)\) and \(T(y)\), it is important to measure the impact of this assumption under the observation that independence is not fulfilled in real life. In the two lives case the maximal impact can be measured using the Fréchet lower and upper bounds introduced in Höffding [35] and Fréchet [36] and first applied to life insurance by Carrière and Chan [6]. Consider the Fréchet class of all bivariate distributions with fixed margins \(q_s = P(T(x) \leq t)\) and \(q_s = P(T(y) \leq t)\). The Fréchet upper bound is the distribution \(F^U(s, t) = P(T(x) \leq s, T(y) \leq t) = \min(q_s, q_s)\) and the Fréchet lower bound is the distribution \(F^L(s, t) = P(T(x) \leq s, T(y) \leq t) = \max(q_s, q_s, 1 - 0)\). Any joint distribution \(F(s, t) = P(T(x) \leq s, T(y) \leq t)\) with fixed margins is constrained from above and below by

\[F^U(s, t) \leq F(s, t) \leq F^L(s, t).\]

The Fréchet bounds generate four different future lifetimes for the joint-life and last-survivor status. Their survival distributions are denoted and determined by
\[ i \bar{p}^L_{xy} = P(T^L(x:y) > t) = \max\{p_x + i_p_y, -1.0\}, \quad (79) \]
\[ i \bar{p}^U_{xy} = P(T^U(x:y) > t) = \min\{p_x, p_y\}, \quad (80) \]
\[ i \bar{p}^L_{xy} = P(T^L(x:y) > t) = \min\{p_x, p_y, 1\}, \quad (81) \]
\[ i \bar{p}^U_{xy} = P(T^U(x:y) > t) = \max\{p_x, p_y\}. \quad (82) \]

For comparison purposes, the future lifetimes and the survival distributions for the joint-life and last-survivor status under the independence assumption are denoted and determined by
\[ i \bar{p}^L_{xy} = P(T^L(x:y) > t) = i\_p_x; p_y, \quad (83) \]
\[ i \bar{p}^U_{xy} = P(T^U(x:y) > t) = i\_p_x + p_y, \quad (84) \]

The defined survival distributions satisfy the inequalities
\[ i \bar{p}^L_{xy} \leq i\_p_x \leq i\_p^U_{xy}, \quad i\_p^U_{xy} \leq i\_p_x \leq i\_p^L_{xy}, \quad (85) \]
which imply stochastic ordering of the corresponding random future lifetimes such that
\[ T^L(x:y) \leq a \quad T^U(x:y) \leq a \quad T^L(x:y), \quad T^U(x:y) \leq a \quad T^L(x:y) \leq a \quad T^U(x:y). \quad (86) \]

According to Section 3, the actuarial reserves as well as the net single and level premiums of \( n \)-year endowment contracts on two lives depend besides survival probabilities on single premiums for life annuities with life status \( (u) \in \{(x), (y), (x:y), (x:y)\} \) of the type
\[ a(u+k:n-k) = \sum_{j=0}^{n-k-1} v^j \cdot p_{uxk}, k = 0,1, \ldots n-1. \quad (87) \]

Inserting the six different life distributions (79)-(84) into (87) and using the stochastic inequalities (85)-(86), one obtains the following bounding inequalities between the different joint-life and last-survivor \( n \)-year life annuities
\[ a^L(x:y:n) \leq a^L(x:y:n) \leq a^L(x:y:n), \quad (88) \]
\[ a^U(x:y:n) \leq a^U(x:y:n) \leq a^U(x:y:n), \quad (89) \]
which imply the following inequalities between the net single premiums (SP) and the net level premiums (P) of \( n \)-year endowment contracts on two lives
\[ SP^L(x:y:n) \leq SP^L(x:y:n) \leq SP^L(x:y:n), \quad (90) \]
\[ P^L(x:y:n) \leq P^L(x:y:n) \leq P^L(x:y:n), \quad (91) \]
\[ SP^L(x:y:n) \leq SP^L(x:y:n) \leq SP^L(x:y:n), \quad (92) \]
\[ P^L(x:y:n) \leq P^L(x:y:n) \leq P^L(x:y:n). \quad (93) \]

To measure further the impact of the independence assumption, it is natural to assume that the future lifetimes \( T(x) \) and \( T(y) \) are positively quadrant dependent and follow a simple Fréchet distribution (Remarks 3.1) defined by
\[ F^\theta(s,t) = (1-\theta) \cdot q_s + \theta \cdot \min\{q_s, q_t\}, \quad \theta \in [0,1], \quad (94) \]
which satisfies the inequality \( F^\theta(s,t) \leq F^\theta(s,t) \leq F^\theta(s,t). \) Using (87) one obtains the following deviations between independence assumption and Fréchet assumption (94) for the net single premiums and level premiums of \( n \)-year endowment contracts on two lives
\[ SP^L(u:n) - SP^\theta(u:n) = iv \cdot \theta \cdot \left( a^L(u:n) - a^\theta(u:n) \right), \quad (95) \]
\[ P^L(u:n) - P^\theta(u:n) = \frac{1}{a^L(u:n)} - \frac{1}{a^\theta(u:n)} + \theta \cdot \left( a^\theta(u:n) - a^L(u:n) \right). \quad (96) \]
Using (88) and (89) it is clear that (95) and (96) are non-negative for the joint-life status \( u = x : y \) and non-positive for the last-survivorship status \( u = x \overline{y} \). Therefore the independence assumption overestimates the joint-life net single and level premiums and underestimates the last-survivor ones. The maximal deviations are obtained for a perfect positive dependence \( \theta = 1 \). These results together with numerical illustration have been presented in [13].

In the following, we investigate whether the independence assumption will similarly underestimate the economic risk capital of a portfolio of last-survivorship endowment contracts and to which extent. For this we use again the simple Fréchet distribution (94). It is important to note by passing that the dependence parameter \( \theta \) can be interpreted as Spearman’s grade correlation coefficient (e.g. [37], Section 3.1). There are not many studies, which provide statistical estimates of this parameter. A strong statistical dependence between female and male mortalities in a married couple has been observed. Based on Frank’s copula with Gompertz marginal distributions, Youn and Shemyakin [8] obtain the estimate \( \theta = 0.54 \). Our marginal future lifetimes follow the distribution of Gompertz [38] (see also Willemse and Kopelaar [39]), which is defined by

\[
p_x = \exp\left\{ \exp\left[ -\left( \frac{a + \beta k}{a + b} \right) \right] (1 - \exp(\frac{a}{b})) \right\}.
\]

The parameters of the Gompertz distribution are set equal to \( a = 85, b = 12 \), for both lived ages \( x \) or \( y \). This choice generates survival probabilities close to those listed in the life table for the 2003 total population in the United States (“National Vital Statistics Reports 54(14), April 19, 2006”, or http://en.wikipedia.org/wiki/Life_table).

Our numerical experience has shown that the pure endowment component strongly determines the life insurance risk of endowment contracts. For this reason and according to the first option made in QIS4 [40], TS.XI.B.3, the survival probabilities in the standard approach of Section 5.1 and in the stochastic approach of Section 5.2 will be shifted with the constant shock of \( \Delta = -0.25 \) for the longevity risk.

To implement the Poisson-Gamma model of Section 6.3, a specification of the annual number of deaths and the remaining cohort sizes beyond the valuation time is required. A simple way, which is consistent with the Solvency II standard approach, consists to assume that future mortality deviates systematically from the life table according to the longevity shock \( \Delta = -0.25 \). For the life aged \( x \) (and similarly for the life aged \( y \)) one sets

\[
\ell_{x_{t+k}} = \ell_{x_{t+k-1}} - d_{x_{t+k-1}}, \quad d_{x_{t+k-1}} = (1+\Delta) \cdot \ell_{x_{t+k-1}} \cdot q_{x_{t+k-1}}, \quad k = 1,2,...
\]

This choice is consistent with the expected number of deaths in the first period \( E[D_{st}] = d_{st} \) if in (70) one sets \( \alpha = (1-\Delta)\beta \). Assume further that \( \beta = 100 \), which implies a coefficient of variation for \( D_{st} \) equal to 10%. One shows that the choice (98) with \( \alpha = (1-\Delta)\beta \) implies that the Poisson-Gamma life table (77) coincides with the shifted life table (59). In this situation the stochastic model of Section 6.3 yields the same results as the method of Section 6.2 and yields a theoretical stochastic justification of the latter. Of course the stochastic model of Section 6.3 is more satisfactory and flexible because it allows the use of effective observed numbers of deaths as time elapses.

While the QIS5 standard solvency ratio does not depend on the initial cohort size, this is the case for the stochastic approaches. We compare solvency ratios of the QIS5 consistent stochastic approaches with the standard approach for the relevant times of valuation \( t \in \{0,1,...,n-2\} \) for cohorts of identical \( n \) year endowment contracts with last-survivorship status \( u = (x:y) \). We use the shifted life table and assume a technical interest rate of 2% and a risk-free interest rate of 3%. The Tables 1 and 2 display shifted coefficients of variation under varying cohort sizes for the ages at entry \( (x,y) \in \{(30,30),(40,40)\} \), the contract term \( n = 10 \), and for the three dependence parameters \( \theta \in \{0,0.5,1\} \) corresponding to the independence assumption, a realistic assumption and the comonotone or perfect dependence assumption. The values are sufficiently small so that the normal approximations (66) are used in the Tables 3 to 6. These Tables display the solvency capital ratios as well as their limiting values (67) for a portfolio of infinitely growing size. For small cohort sizes and early times of valuation, the QIS5 standard approach prescribes insufficient almost vanishing solvency ratios. In fact, as already explained, it does not take into account the process risk. On the other hand, solvency ratios of cohort sizes exceeding 10’000 contracts are close to those of the standard approach and tend more and more to the limited value, as expected. Moreover, the independence assumption clearly underestimates the economic risk capital of a portfolio of last-survivorship endowment contracts The Figure 1 displays these findings in a visual way for the safest perfect dependence assumption \( \theta = 1 \).
Table 1. Coefficients of variation of the shifted random present value of future cash-flows, \((x, y) = (30,30)\)

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<tr>
<th>Cohort size</th>
<th>Time of Valuation</th>
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<th>1</th>
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<tr>
<td>100</td>
<td>0.143% 0.064% 0.038% 0.024% 0.016% 0.011% 0.007% 0.004% 0.002%</td>
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<tr>
<td>500</td>
<td>0.064% 0.029% 0.017% 0.011% 0.007% 0.005% 0.003% 0.002% 0.001%</td>
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<tr>
<td>1'000</td>
<td>0.045% 0.020% 0.012% 0.008% 0.005% 0.003% 0.002% 0.001% 0.000%</td>
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<td>10'000</td>
<td>0.014% 0.006% 0.004% 0.002% 0.002% 0.001% 0.001% 0.000% 0.000%</td>
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<td>100'000</td>
<td>0.005% 0.002% 0.001% 0.001% 0.001% 0.000% 0.000% 0.000% 0.000%</td>
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<tr>
<td>100</td>
<td>3.563% 1.557% 0.889% 0.556% 0.357% 0.227% 0.138% 0.074% 0.031%</td>
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<td>500</td>
<td>1.593% 0.697% 0.398% 0.248% 0.160% 0.102% 0.062% 0.033% 0.014%</td>
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<tr>
<td>1'000</td>
<td>1.127% 0.493% 0.281% 0.176% 0.113% 0.072% 0.044% 0.024% 0.010%</td>
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<td>10'000</td>
<td>0.356% 0.156% 0.089% 0.056% 0.036% 0.023% 0.014% 0.007% 0.003%</td>
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<td>5.041% 2.204% 1.258% 0.786% 0.505% 0.321% 0.195% 0.105% 0.043%</td>
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<tr>
<td>500</td>
<td>2.255% 0.985% 0.562% 0.351% 0.226% 0.144% 0.087% 0.047% 0.019%</td>
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<td>1'000</td>
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<td>10'000</td>
<td>0.504% 0.220% 0.126% 0.079% 0.051% 0.032% 0.019% 0.011% 0.004%</td>
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<tr>
<td>100'000</td>
<td>0.159% 0.070% 0.040% 0.025% 0.016% 0.010% 0.006% 0.003% 0.001%</td>
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Table 2. Coefficients of variation of the shifted random present value of future cash-flows, \((x, y) = (40,40)\)

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<th>Cohort size</th>
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<td>100</td>
<td>0.329% 0.148% 0.087% 0.056% 0.037% 0.024% 0.015% 0.008% 0.004%</td>
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<td>500</td>
<td>0.147% 0.066% 0.039% 0.025% 0.017% 0.011% 0.007% 0.004% 0.002%</td>
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<td>1'000</td>
<td>0.104% 0.047% 0.027% 0.018% 0.012% 0.008% 0.005% 0.003% 0.001%</td>
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<td>10'000</td>
<td>0.033% 0.015% 0.009% 0.006% 0.004% 0.002% 0.002% 0.001% 0.000%</td>
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<td>0.010% 0.005% 0.003% 0.002% 0.001% 0.001% 0.000% 0.000% 0.000%</td>
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<tr>
<td>100</td>
<td>5.415% 2.367% 1.351% 0.844% 0.543% 0.345% 0.209% 0.113% 0.047%</td>
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<tr>
<td>500</td>
<td>2.422% 1.059% 0.604% 0.378% 0.243% 0.154% 0.093% 0.051% 0.021%</td>
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<td>1'000</td>
<td>1.712% 0.749% 0.427% 0.267% 0.172% 0.109% 0.066% 0.036% 0.015%</td>
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<td>10'000</td>
<td>0.542% 0.237% 0.135% 0.084% 0.054% 0.035% 0.021% 0.011% 0.005%</td>
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<td>100'000</td>
<td>0.171% 0.075% 0.043% 0.027% 0.017% 0.011% 0.007% 0.004% 0.001%</td>
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<tr>
<td>100</td>
<td>7.668% 3.351% 1.912% 1.195% 0.768% 0.488% 0.295% 0.160% 0.066%</td>
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<tr>
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<tr>
<td>1'000</td>
<td>2.425% 1.060% 0.605% 0.378% 0.243% 0.154% 0.093% 0.050% 0.021%</td>
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<tr>
<td>10'000</td>
<td>0.767% 0.335% 0.191% 0.119% 0.077% 0.049% 0.030% 0.016% 0.007%</td>
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<td>100'000</td>
<td>0.242% 0.106% 0.060% 0.038% 0.024% 0.015% 0.009% 0.005% 0.002%</td>
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### Table 3. VaR solvency capital ratios for last-survivorship endowment contracts, \((x, y) = (30, 30)\)

<table>
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<tr>
<th>Cohort size</th>
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### Table 4. VaR solvency capital ratios for last-survivorship endowment contracts, \((x, y) = (40, 40)\)

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### Table 5. CVaR solvency capital ratios for last-survivorship endowment contracts, \((x, y) = (30, 30)\)

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### Table 6. CVaR solvency capital ratios for last-survivorship endowment contracts, \((x, y) = (40, 40)\)

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Figure 1. Time evolution of VaR solvency capital ratios, \((x, y) = (40, 40), \theta = 1\)

7. REFERENCES


