ADOMIAN DECOMPOSITION METHOD AND TAYLOR SERIES METHOD IN ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT
In this paper, it is revealed that Adomian decomposition method corresponds to Taylor series method when applied to the solution of nonlinear initial value problems, in the following sense: the Adomian´s polynomials can be obtain trough Taylor coefficients.

Keywords: *Taylor series method, Adomian decomposition method, nonlinear differential equations.*

1. INTRODUCTION

There are many new analytical approximate methods to solve initial value problems in the literature. Among these, Adomian´s decomposition method [2-3] have been received much attention in recent years in applied mathematics in general, and easily handle a wide a class of linear or nonlinear problems.
The Adomian technique is based on a decomposition of a solution of a nonlinear functional equation in a series of functions. Each term of the series is obtained from a polynomial generated by a power series expansion of an analytic function.
The main advantage of the ADM is that it can be applied directly for all types of functional equations, linear or nonlinear. Another important advantage is capable of greatly reducing the size of computation work while still maintaining high accuracy of the numerical solution.
In [9] the author compared the Adomian Decomposition Method (ADM) and the Taylor series method by using some particular examples, and showed that the Adomian´s technique produced reliable results with a fewer iterations, whereas the Taylor series method suffered from computational difficulties. But in this paper, we will show that both Adomian decomposition method and Taylor series method are equivalents and therefore their convergence is the same in both.
The following result will be useful in this paper: Faà di Bruno formula gives an explicit equation for the nth derivative of the composition f(g(t)) . Such result will be used in all work.

Theorem (Faà di Bruno formula set-partition version)
Let $n \in N$ and f and g functions with a sufficient number of derivatives, then

$$D^n f(g(x)) = \sum_{n=0}^{\infty} \frac{n!}{k_1!k_2!\cdots k_n!} \left(D^k f\right)\left(g(t)\right)\left(\frac{Dg(t)}{1!}\right)^{k_1}\left(\frac{D^2 g(t)}{2!}\right)^{k_2}\cdots\left(\frac{D^n g(t)}{n!}\right)^{k_n}$$

Where $k = k_1 + k_2 + \cdots + k_n$ and the last sum is over all partitions of n, i.e., values of $k_1, k_2, \cdots, k_n$ such that $k_1 + 2k_2 + \cdots + nk_n = n$.
The proof of this formula is found in [7].

2. ADOMIAN DECOMPOSITION METHOD

The Adomian´s technique depends on decomposing the nonlinear differential equation

$$F(x, u(x), u'(x)) = 0 \qquad (1)$$

Into the two components

$$L(u(x)) + N(u(x)) = 0$$

where L and N are the linear and nonlinear parts of F respectively. The operator L is assumed to be an invertible operator. Solving for L(u) leads to

$$L(u(x)) = -N(u(x)) \qquad (2)$$

Appling the inverse operator L^{-1} on both sides of Eq. (2) yields

$$u(x) = \varphi(x) - L^{-1}[N(u(x))] \qquad (3)$$

where $\varphi(x)$ is the constant of integration satisfies the condition $L(\varphi) = 0.$ Now assuming that the solution y can be represented as infinite series of the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (4)$$

Furthermore, suppose that nonlinear term $N(u)$ can be written as infinity series in terms of the Adomian polynomials A_n of the form

$$N(u) = \sum_{n=0}^{\infty} A_n \quad , \qquad (5)$$

where A_n are Adomian polynomials of $u_0, u_1, \ldots, u_n$ (see [1-6]) is given by following formula:

$$A_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{n} u_i \lambda^i\right)\right]_{\lambda=0} \qquad (6)$$

Then substituting Eqs. (4) and (5) in Eq.(3) gives

$$\sum_{n=0}^{\infty} u_n(x) = \varphi(x) - L^{-1}\left(\sum_{n=0}^{\infty} A_n(s)\right); \qquad (7)$$

this gives the recurrence scheme of ADM as

$$\begin{cases} u_0(x) = \varphi(x) \\ u_{k+1}(x) = -L^{-1}(A_k), \quad k \geq 0. \end{cases} \qquad (8)$$

in order to obtain the Adomian´s polynomials, the following algorithm will be used (see [3])

$$A_0(x) = N(y_0)$$

$$A_1(x) = \frac{d}{d\lambda} N(y_0 + \lambda y_1)\Big|_{\lambda=0} = y_1(x) N'(y_0)$$

$$A_2(x) = \frac{1}{2!}\frac{d^2}{d\lambda^2} N(y_0 + \lambda y_1 + \lambda^2 y_2)\Big|_{\lambda=0} = y_2(x) N'(y_0) + \frac{1}{2} y_1^{\,2} N''(y_0)$$

$$A_3(x) = \frac{1}{3!}\frac{d^3}{d\lambda^3} N\left(\sum_{n=0}^{3} \lambda^n y_n(x)\right)\Big|_{\lambda=0} = y_3(x) N'(y_0) + y_1(x) y_3(x) + \frac{1}{3!} y_1^{\,3} N'''(y_0) \qquad (9)$$

$$A_3(x) = \frac{1}{4!}\frac{d^4}{d\lambda^4} N\left(\sum_{n=0}^{4} \lambda^n y_n(x)\right)\Big|_{\lambda=0} = y_4(x) N'(y_0) + \left(y_1(x) y_3(x) + \frac{1}{2} y_2^{\,2}(x)\right) N''(y_0) +$$

$$+ \frac{1}{2} y_1^{\,2}(x) y_2(x) N'''(y_0) + \frac{1}{4!} y_1^{\,4}(x) N^{(4)}(y_0)$$

$$\vdots$$

When we tried to solve the equation in analytical form, the process is longer. However, in practice all the terms of series (7) cannot be determined, and the solution is approximated by the truncated series $\sum_{n=0}^{N} u_n$. Although in many situations you can found the solution in closed form using nested integrals properties, see for example [8].

Proposition

Consider the differential equation

$$\frac{du}{dx} = N(u(x)), \qquad (10)$$

together the initial condition

$$u(x_0) = u_0. \qquad (11)$$

Then, the general solution gives by Taylor´s series method

$$u(x) = u(x_0) + \frac{u'(x_0)}{1!}(x - x_0) + \frac{u''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{u^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$

is precisely the Adomian Decomposition Method:

$$u(x) = u_0 + u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

where $u_k(x) := \frac{u^{(k)}(x_0)}{k!}(x - x_0)^k$, $k = 0,1,2,\cdots$ and u_k , $k = 0,1,\cdots$ come determined by the iterative scheme:

$$\begin{cases} u_0 = u(x_0) \\ u_n(x) = \int_{x_0}^{x} A_{n-1}(s)ds \quad , n = 1,2,3,\cdots \end{cases} \qquad (12)$$

where A_k , $k = 0,1,\cdots$ verifies

$$A_k(x) = \frac{1}{k!}\frac{d^k}{dx^k}\left[N(u(x))\right]_{x=x_0}(x - x_0)^{k-1}, \qquad k = 1,2,\cdots \qquad (13)$$

Proof: Replacing the initial condition (11) into Eq. (10) one gets $u'(x_0) = f(x_0, u_0) = N(u_0)$ so that, $A_0 = u'(x_0)$.

Now, by derivation Eq. (10) with respect x, one gets

$$u''(x) = \frac{d}{dx}\left[N(u(x))\right] = N'(u(x))u'(x), \qquad (14)$$

by using the initial conditions: $u(x_0) = u_0$ and $u'(x_0) = N(u_0)$ in Eq. (14) we obtain $u''(x_0) = N'(u_0)u'(x_0)$.

Then, by multiplying $(x - x_0)$ both sides of above equation and by using the above notation, we have

$$\frac{u''(x_0)}{1!}(x - x_0) = \left(\frac{u'(x_0)}{1!}(x - x_0)\right)N'(u_0) = u_1(x)N'(u_0) = A_1(x)$$

Now, the next step is integrate to both sides the last equation $\int_{x_0}^{x} \frac{u''(x_0)}{1!}(s - x_0)ds = \int_{x_0}^{x} A_1(s)ds$

That is, since $\int_{x_0}^{x} \frac{u''(x_0)}{1!}(s - x_0)ds = \frac{u''(x_0)}{2!}(s - x_0)^2 = u_2(x)$, we have $u_2(x) = \int_{x_0}^{x} A_1(s)ds$

By derivation Eq. (14) again, we obtain

$$u'''(x) = \frac{d^2}{dx^2}\left[N(u(x))\right] = N''(u(x))\left[u'(x)\right]^2 + N'(u(x))u'(x)u''(x), \qquad (15)$$

.

Replacing $x = x_0$ in Eq. (15) and divide by 2!, and by multiplying to both sides of equation for $(x - x_0)^2$, we have

$$\frac{1}{2!}u'''(x_0)(x-x_0)^2 = \frac{1}{2!}\left[N''(u(x_0))[u'(x_0)]^2 + N'(u(x_0))u'(x_0)u''(x_0)\right](x-x_0)^2$$

$$= \frac{1}{2!}[u'(x_0)(x-x_0)]^2 N''(u_0) + \left[\frac{u''(x_0)}{2!}(x-x_0)^2\right]N'(u_0)$$

Therefore, by using the notation

$$\frac{1}{2!}u'''(x_0)(x-x_0)^2 = \frac{1}{2!}u_1^{\,2}(x)N''(u_0) + u_2(x)N'(u_0) = A_2(x)$$

By integration to both sides of above equation, we obtain

$$u_3(x) = \frac{u'''(x_0)}{3!}(x-x_0)^3 = \int_{x_0}^{x}\frac{u'''(x_0)}{2!}(s-x_0)^2\,ds = \int_{x_0}^{x}A_2(s)ds$$

Then,

$$u_3(x) = \int_{x_0}^{x}A_2(s)ds$$

By continuing of the same way this process, one gets

$$\frac{u^{(n+1)}(x_0)}{n!}(x-x_0)^n = \frac{1}{n!}\frac{d^n}{dx^n}\left[N(u(x))\right]_{x=x_0}(x-x_0)^n$$

$$= A_n(x)$$

Integrate both sides of above equation, we have

$$u_n(x) = \int_{x_0}^{x}A_{n-1}(s)ds$$

The above iterative scheme has been constructed:

$$\begin{cases} u_0 = u(x_0) \\ u_n(x) = \int_{x_0}^{x}A_{n-1}(s)ds \quad , n = 1,2,3,\cdots \end{cases}$$

Where $A_i(x), \quad i = 0,1,2,\cdots$ verifies

$$A_k(x) = \frac{1}{k!}\frac{d^k}{dx^k}\left[N(u(x))\right]_{x=x_0}(x-x_0)^{k-1}, \qquad k = 1,2,\cdots$$

Then the general solution of initial value problem Eqs. (10) -(11) becomes $u(x) = u_0 + u_1 + u_2 + \cdots + u_n + \cdots$
That is, the Adomian decomposition method.

3. APPLICATIONS AND RESULTS: A LINEAR EXAMPLE

Since the problems given in the article at reference [9] (Wazwaz, 1998) on the computations and analysis of Taylor series method are relevant for our aim, we reuse some of the problems.

Consider the linear initial value problem

$$e^x u'' + xu = 0\,, \tag{16}$$

subject to the initial conditions

$$u(0) = A, \quad u'(0) = B. \tag{17}$$

The comparison will be made by applying the two methods separately.

3.1 The Taylor series method

The Taylor series method introduces the solution by an infinite series given by

$$u(x)=\sum_{n=0}^{\infty} a_n x^n. \qquad (18)$$

In [9] the author replaces Eq. (17) into Eq. (16) to obtain

$$\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\left(\sum_{n=2}^{\infty} n(n+1)a_n x^{n-2}\right)+\sum_{n=0}^{\infty} a_n x^{n+1}=0$$

The coefficients $a_n,\, n\geq 0$, are determined by equating coefficients of like powers of x through determining a formal recurrence relation. The author shows that an explicit recurrence relation is difficult, alternatively, term a term he multiplies the series involved.

Instead of using the solution proposed by [9], i.e., multiplying the exp. series by solutions series, we consider the following steps:

Substituting the initial condition on Eqs. (17) in Eq. (16) we obtain $u''(0)=0.$

By derivation Eq. (16) yields

$$e^x u'''(x)+e^x u'''(x)+xu'(x)+u(x)=0\,, \qquad (19)$$

Replacing $u(0)=A, u'(0)=B$ and $u''(0)=0$ into Eq. (19) we have $u'''(0)=-A.$

Continuing this process we obtain the first derivatives at $x=0$:

$$u(0)=A,\; u'(0)=B,\; u''(0)=0,\; u'''(0)=-A,\quad u^{(iv)}(0)=2A-2B,\quad u^{(v)}(0)=6B-3A,\cdots$$

Then, the approximate solution is given by

$$\begin{aligned}
u(x) &\approx u(0)+\frac{u'(0)}{1!}x+\frac{u''(0)}{2!}x^2+\frac{u'''(0)}{3!}x^3+\cdots+\frac{u^{(n)}(0)}{n!}x^n+\cdots\\
&= A+Bx-\frac{A}{3!}x^3+\frac{(2A-2B)}{4!}x^4+\frac{(6B-3A)}{5!}x^5+\cdots\\
&= A\left(1-\frac{1}{6}x^3+\frac{1}{12}x^4-\frac{1}{40}x^5+\cdots\right)+B\left(1-\frac{1}{12}x^4+\frac{1}{20}x^5+\cdots\right) \qquad (20)
\end{aligned}$$

3.2 The Adomian decomposition method

According [7,9], Eq. (16) can be written in an operator form as

$$L_{xx}u=-xe^{-x}u \qquad (21)$$

Where $L_{xx}(.)=\frac{d^2}{dx^2}(.)$. Then inverse of L is, therefore, $L^{-1}{}_{xx}(.)=\int_0^x\int_0^s(.)duds$. Applying L^{-1} to both sides of (21) we find that

$$u(x)=A+Bx-\int_0^x\int_0^s we^{-w}u(w)dwds \qquad (22)$$

The decomposition method consists of decomposing $u(x)$ into a sum of components given by the infinite series

$$u(x)=\sum_{n=0}^{\infty} u_n. \qquad (23)$$

Substituting Eq. (23) into both sides of Eq.(22) yields

$$\sum_{n=0}^{\infty} u_n(x)=A+Bx-\int_0^x\left(\int_0^s\left[we^{-w}\sum_{n=0}^{\infty}u_n(w)dw\right]\right)ds \qquad (24)$$

Next, we equate selected components on both sides using the following recursive relationship:

$$u_0 = A + Bx$$

$$u_{k+1} = -L^{-1}{}_{xx}\left(xe^{-x}\sum_{n=0}^{\infty} u_k(x)\right), \quad (k \geq 0).$$

According, we find

$$u_0 = A + Bx$$

$$u_1 = -L^{-1}{}_{xx}\left(xe^{-x}u_0\right) = -L^{-1}{}_{xx}\left(A\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{n+1} + B\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{n+2}\right)$$

$$= A\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+3)(n+2)n!}x^{n+3} - B\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+4)(n+3)n!}x^{n+4}$$

$$= A\left(\frac{1}{6}x^3 - \frac{1}{212}x^4 + \frac{1}{40}x^3 + \cdots\right) + B\left(\frac{1}{12}x^4 - \frac{1}{20}x^5 + \frac{1}{60}x^6 + \cdots\right)$$

Which again gives the solution obtained in Eq.(20).

4. APPLICATION: A NONLINEAR EXAMPLE

Consider the nonlinear initial value problem

$$\frac{dy}{dx} = \frac{y^2}{1-xy}, \quad y(0) = 1. \qquad (25)$$

In the same manner as was done in example 1, we will compare this example with the two methods.
We initially assume the function y(x) supports derivatives of all orders.

4.1 The Taylor series method

In [1] the author replaces Eq. (23) into Eq. (25) to obtain

$$\sum_{n=1}^{\infty} na_n x^{n-1} = \left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\left(\sum_{n=0}^{\infty} na_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right)^2 \qquad (26)$$

The author shows that Eq. (26) presents computational difficulties, instead of using the solution proposed by [9], i.e., multiplying the exponential series by solutions series, we consider the following steps:
Eq. (25) is equivalent to equation

$$(1-xy)y' = y^2, \qquad y(0) = 1, \quad (xy \neq 1) \qquad (27)$$

Substituting the initial condition $y(0) = 1$ in Eq. (27) we obtain $y'(0) = 1$.
By derivation Eq. (27) yields

$$y''(1-xy) - y'(3y + xy') = 0, \qquad (28)$$

Replacing $y(0) = 1,\ y'(0) = 1$ in Eq. (28) we have $y''(0) = 3$.
By derivation again Eq. (28) on gets

$$y'''(1-xy) - y''(4y + 3xy') - 4(y')^2 = 0, \qquad (29)$$

Replacing the values $y(0) = 1,\ y'(0) = 1,\ y''(0) = 3$ in Eq. (29) one gets $y'''(0) = 16$.
By derivation once more

$$y^{(iv)}(1-xy) - y'''(5y + 4xy') - y''(15y' + 3xy'') = 0, \qquad (30)$$

Replacing the values $y(0) = 1,\ y'(0) = 1,\ y''(0) = 3$ and $y'''(0) = 16$ in Eq. (30) one gets $y^{(iv)}(0) = 125$.
Then, the approximate solution is given by

$$y(x)\approx y(0)+\frac{y'(0)}{1!}x+\frac{y''(0)}{2!}x^2+\frac{y'''(0)}{3!}x^3+\cdots+\frac{y^{(n)}(0)}{n!}x^n+\cdots$$
$$=1+x+\frac{3}{2}x^2+\frac{8}{3}x^3+\frac{125}{4!}x^4+\cdots$$

Again we can calculate recurrently so many coefficients in the series as necessary to produce a solution with a desired accuracy for all real x sufficiently close x=0. However we cannot express the general coefficient a_n of the series by means of an explicit function of n and therefore, we cannot directly calculate the radius of convergence of this solution. Such situations occur when you are looking for solutions in series of powers of nonlinear equations.

4.2 Adomian decomposition method

Again, we can put Eq. (25) in an operator form as

$$L_x y = xyy' + y^2,\quad y(0)=1, \qquad (31)$$

In this case, $L_x(.)=\frac{d}{dx}(.)$ is the ordinary differential operator, and $L^{-1}{}_x(.)=\int_0^x(.)ds$ is an integral operator. Applying as before $L^{-1}{}_x$ to both sides of Eq. (31) and using the initial conditions we find

$$y(x)=1+L^{-1}{}_x(xyy')+L^{-1}{}_x(y^2). \qquad (32)$$

The decomposition method expands each of the nonlinear terms yy' an y^2 formally in a power series, given by

$$yy'=\sum_{n=0}^{\infty}A_n,\qquad y^2=\sum_{n=0}^{\infty}B_n,$$

were A_n and B_n are the so-called Adomian polynomials corresponding to the nonlinear terms yy' an y^2, respectively. The Adomian polynomials are given by

$$A_0=y_0y_0',\quad A_1=y_0'y_1+y_1'y_0,\quad A_2=y_0'y_2+y_1'y_1+y_2'y_0,\quad\cdots \qquad (33)$$

and

$$B_0={y_0}^2,\quad B_1=2y_0'y_1,\quad B_2=2y_0y_2+{y_1}^2,\quad\cdots \qquad (34)$$

Substituting Eqs. (4), (33) and (34) into the functional equation (32) gives

$$\sum_{n=0}^{\infty}y_n=1+L^{-1}{}_x\left(\sum_{n=0}^{\infty}xA_n\right)+L^{-1}{}_x\left(\sum_{n=0}^{\infty}B_n\right). \qquad (35)$$

Here we can assume the convergence of the series, both sides of Eq. (35) will match by setting the recursive relationship

$$y_0(x)=1$$
$$y_{k+1}(x)=L_x^{-1}(xA_k)+L_x^{-1}(B_k)\quad,\quad k\ge 0$$

This leads to

$$y_0(x)=1,\quad y_1(x)=L_x^{-1}(xA_0)+L_x^{-1}(B_0)=x,\quad y_2(x)=L_x^{-1}(xA_1)+L_x^{-1}(B_1)=\frac{3}{2}x^2,$$
$$y_3(x)=L_x^{-1}(xA_2)+L_x^{-1}(B_2)=\frac{8}{3}x^3,\quad y_4(x)=L_x^{-1}(xA_3)+L_x^{-1}(B_3)=\frac{125}{4}x^4,\cdots$$

The solution of Eq.(25) in a series form is therefore

$$y(x)=1+x+\frac{3}{2}x^2+\frac{8}{3}x^3+\frac{125}{4!}x^4\cdots$$

The next example shows that the Adomian decomposition is more complicated than Taylor series method, because its polynomial cannot found easily, but one can use the coefficients found by Taylor and then by using Adomian method can find the solution. Consider the initial value problem

$$u'(t)=\frac{u}{t-u},\qquad u(0)=1,$$

This problem has exact solution: $u\ln u + t = 0, \quad t < \frac{1}{e}.$

Proceeding as before, we write the differential equation in the form:

$$u'(t-u) = u, \quad u'(0) = -1$$

By repeating the process carried out with the previous examples. Doing the successive derivatives we found that

$$u''(t) = \frac{u^2}{(t-u)^3}, \qquad u''(0) = -1,$$

$$u'''(t) = \frac{u^2}{(t-u)^5}(4u - t), \qquad u'''(0) = -4,$$

$$u^{(4)}(t) = \frac{u^2}{(t-u)^5}\left(27u^2 - 14ut + 2t^2\right), \qquad u^{(4)}(0) = -27,$$

$$\vdots$$

$$u^{(n)}(t) = \frac{u^2}{(t-u)^{2n-3}}\left((n-1)^{(n-1)}u^{n-2} + P_{n-2}(t,u)\right), \qquad u^{(n)}(0) = -(n-1)^{(n-1)}, \; n \geq 2$$

So, the solution in Taylor series is therefore:

$$u(t) = 1 - \sum_{n=1}^{\infty} \frac{n^n}{(n+1)!} t^{n+1},$$

We find that the radius of convergence of the series is given by

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{an}\right| = |t| \lim_{n\to\infty}\left|\frac{(n+1)^{(n+1)}}{(n+2)!}\,\frac{(n+1)}{n^n}\right| = |t|\lim\left(1+\frac{1}{n}\right)^n < 1 \Rightarrow \quad |t| < \frac{1}{e}$$

This same problem with a best result has been treated in [5].

Although, the decomposition method provides the same answer obtained by the Taylor method, it involves more computational work. In addition, a recurrence relation was not easy to obtain by using Adomian method.

We conclude that although, the Adomian decomposition method provides the same answer obtained by Taylor series with the same computational work. We should to be patient.

5. CONCLUSION

The Adomian decomposition method, after all continue to be a good method for solving nonlinear initial value problems; it should be noted that certain nonlinear problems can be solved analytically by using this method and certain properties of integration, such as nested integral [8], whereas Taylor series solution would approximate solution.

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