NEW ALGORITHM FOR SOLVING THE MODIFIED KORTEWEG-DE VRIES (mKdV) EQUATION

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ABSTRACT
In this paper we will investigate the finite element approach using collocation method with septic spline interpolation functions to solve the Modified Korteweg-de Vries (mKdV) equation. It is unconditionally stable by von Neumann method. The exact soliton solution and the conserved quantities are used to assess the accuracy and to show the robustness of the scheme. The interaction of two solitary waves for different parameters is discussed.

AMS classification: 65N30; 65N35; 65D07; 76B25
Keywords: Finite elements; Collocation; splines; Solitary waves; MKdV

1. INTRODUCTION
In this paper we will introduce a numerical solution for the Modified Korteweg- de Vries equation (mKdV) which is a non-linear partial differential equation which involves both convection and diffusion, in which numerical schemes are the most available method to study its solutions. In this work we introduce the septic spline with finite element method to solve the MKdV equation, and discuss the stability and the accuracy of this solution comparing with the exact solution with some initial and boundary conditions.

2. THE MKdV EQUATION
The MKdV equation in the x-direction of the form [1,4,11]
\[ u_t + 
\alpha u^2 u_x + 
\beta u_{xxx} = 0, \quad a \leq x \leq b \]
Where \( \alpha, \beta \) are positive parameters and the subscripts \( x \) and \( t \) denote differentiation w.r.t. \( x \) and \( t \) respectively, with the physical boundary conditions
\[ u(a,t) = u(b,t) = 0, u_x(a,t) = u_x(b,t) = 0, \quad (1) \]
in addition to the initial condition \( u(x,0) = f(x) \).

3. COLLOCATION WITH SEPTIC SPLINES
Let \( a = x_0 \leq \ldots \leq x_n = b \) be an equipartitioned \( (n+1) \) points for \( [a,b] \), it is known that the set \( B_m(x), m = -3(1)N+3 \) form a basis for the space of solution at the collocation points (the knots) [9], be given as

\[
B_m(x) = \frac{1}{h^5} \begin{cases}
\{-x_m^4 - 8 (x_m - x_{m-1})^7, & x_{m-1} \leq x \leq x_m \\
\{x_m^4 - x_{m-1}^7 - 8 (x - x_m - 1)^7, & x_{m-1} \leq x \leq x_m \\
\{x_{m+1}^2 - x_{m+1} - 1 \\
\{x_{m+2}^2 - x_{m+2} - 1 \\
\{x_{m+3}^2 - x_{m+3} - 1 \\
\{x_{m+4}^2 - x_{m+4} - 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[ m \leq x \leq x_{m-4} \]
\[ x_{m-4} \leq x \leq x_{m-3} \]
\[ x_{m-3} \leq x \leq x_{m-2} \]
\[ x_{m-2} \leq x \leq x_{m-1} \]
\[ x_{m-1} \leq x \leq x_m \]
\[ x_m \leq x \leq x_{m+1} \]
\[ x_{m+1} \leq x \leq x_{m+2} \]
\[ x_{m+2} \leq x \leq x_{m+3} \]
\[ x_{m+3} \leq x \leq x_{m+4} \]
Where the values of the septic splines $B_m(x)$, and all its first and second derivatives vanishes outside the interval $[x_m, x_{m+4}]$.

Our task is to find an approximate solution $U_N(x,t)$ to the solution $u(x,t)$ in the form:

$$U_N(x,t) = \sum_{m=3}^{N+3} C_m(t)B_m(x)$$

(4)

Where $C_m(t)$ are unknown time dependent parameters which are determined from collocation boundary and initial conditions. The nodal values $U_m, U'_m, U''_m, U'''_m$ [2,3,5,6,7] at the notes $x_m$ are obtained from Eqs. (3) and (4) in the following form

$$U_m = U(x_m) = c_{m-3} + 120c_{m-2} + 1191c_{m-1} + 2416c_m + 1191c_{m+1} + 120c_{m+2} + c_{m+3}$$

$$U'_m = U'(x_m) = \frac{7}{h}(-c_{m-3} - 56c_{m-2} - 245c_{m-1} + 245c_{m+1} + 56c_{m+2} + c_{m+3})$$

$$U''_m = U''(x_m) = \frac{42}{h^2}(c_{m-3} + 24c_{m-2} + 15c_{m-1} - 80c_m + 15c_{m+1} + 24c_{m+2} + c_{m+3})$$

$$U'''_m = U'''(x_m) = \frac{210}{h^3}(-c_{m-3} - 8c_{m-2} + 19c_{m-1} - 19c_{m+1} + 8c_{m+2} + c_{m+3})$$

(5)

The time derivative of Eq. (1) is discretized by a first order accurate forward difference formula and by using the $\theta$-weighted ($0 \leq \theta \leq 1$), scheme to the space derivative at two time levels to get the equation

$$\frac{U^{n+1} - U^n}{k} + \epsilon(\theta(U^2U_x)^{n+1} + (1 - \theta)(U^2U_x)^n) + \mu(\theta(U_{xxx})^{n+1} + (1 - \theta)(U_{xxx})^n)$$

(6)

Where $k$ is the time step and the superscripts $n$ and $n+1$ are successive time levels. In this work we take $\theta = 0.5$, hence Eq. (6) is written as

$$\frac{U^{n+1} - U^n}{k} + \epsilon\left(\frac{U^2U_{x}^{n+1} + (U^2U_{x})^n}{2}\right) + \mu\left(\frac{U_{xxx}^{n+1} + (U_{xxx})^n}{2}\right)$$

(7)

The nonlinear term in Eq. (7) is approximated by applying Taylor series [10]

$$(U^2U_x)^{n+1} \approx (U^2U_x)^n + (U^2)^{n+1}(U_x)^n - (U^2U_x)^n$$

(8)

At the nth time step, we denote $U, U', U'', U'''$ at the notes $x_m$ by the following expressions

$$L_{m1} = c_{m-3} + 120c_{m-2} + 119c_{m-1} + 2416c_m + 119c_{m+1} + 120c_{m+2} + c_{m+3}$$

$$L_{m2} = \frac{7}{h}(-c_{m-3} - 56c_{m-2} - 245c_{m-1} + 245c_{m+1} + 56c_{m+2} + c_{m+3})$$

$$L_{m3} = \frac{42}{h^2}(c_{m-3} + 24c_{m-2} + 15c_{m-1} - 80c_m + 15c_{m+1} + 24c_{m+2} + c_{m+3})$$

$$L_{m4} = \frac{210}{h^3}(-c_{m-3} - 8c_{m-2} + 19c_{m-1} - 19c_{m+1} + 8c_{m+2} + c_{m+3})$$

(9)

Using the knots $x_m, m = 0, 1, 2, ..., N$ as collocation points, the following recurrence relation at point $x_m$ is obtained using Eqs. (6)-(9)
\begin{equation}
z_1 c_{m-3}^{n+1} + z_2 c_{m-2}^{n+1} + z_3 c_{m-1}^{n+1} + z_4 c_m^n + z_5 c_{m+1}^{n+1} + z_6 c_{m+2}^{n+1} + z_7 c_{m+3}^{n+1} = 2h^3 (L_m + \frac{\varepsilon k}{2} L_m^2 L_m^2 - \frac{\mu k}{2} L_m^3) \tag{10}
\end{equation}

Where

\begin{align*}
z_1 &= L_m^0 - 7\varepsilon k h^3 L_m^2 - 210 \mu k, \\
z_2 &= 120 L_m^0 - 392 \varepsilon k h^3 L_m^2 - 1680 \mu k, \\
z_3 &= 1191 L_m^0 - 1715 \varepsilon k h^3 L_m^2 + 3990 \mu k, \\
z_4 &= 2416 L_m^0, \\
z_5 &= 1191 L_m^0 + 1715 \varepsilon k h^3 L_m^2 - 3990 \mu k, \\
z_6 &= 120 L_m^0 + 392 \varepsilon k h^3 L_m^2 + 1680 \mu k, \\
z_7 &= L_m^0 + 7\varepsilon k h^3 L_m^2 + 210 \mu k, \\
L_m^0 &= 2h^3 (1 + \varepsilon k L_m^2 L_m^2), \text{ where } m = 0, 1, 2, ..., N
\end{align*} \tag{11}

The system \((10)\) consists of \((N+1)\) equations in \((N+7)\) unknowns \(c_{-3}, c_{-2}, c_{-1}, c_0, ..., c_{N+1}, c_{N+2}, c_{N+3}\). To obtain a unique solution to this system we need six additional constraints. These are obtained from the boundary conditions (2) which require that:

\begin{align*}
c_{-3} + 120 c_{-2} + 1191 c_{-1} + 2416 c_0 + 1191 c_1 + 120 c_2 + c_3 &= 0, \\
-7c_{-3} - 392 c_{-2} - 1715 c_{-1} + 1715 c_1 + 392 c_2 + 7c_3 &= 0, \\
42 c_{-3} + 1008 c_{-2} + 630 c_{-1} - 3360 c_0 + 630 c_1 + 1008 c_2 + 42 c_3 &= 0, \\
c_{N-3} + 120 c_{N-2} + 1191 c_{N-1} + 2416 c_N + 1191 c_{N+1} + 120 c_{N+2} + c_{N+3} &= 0, \\
-7c_{N-3} - 392 c_{N-2} - 1715 c_{N-1} + 1715 c_{N+1} + 392 c_{N+2} + 7c_{N+3} &= 0, \\
42 c_{N-3} + 1008 c_{N-2} + 630 c_{N-1} - 3360 c_N + 630 c_{N+1} + 1008 c_{N+2} + 42 c_{N+3} &= 0
\end{align*} \tag{12}

Eliminating \(c_{-3}, c_{-2}, c_{-1}, c_N, c_{N+1}, c_{N+2}\) and \(c_{N+3}\) from our scheme \((10)\), then \((N+1)\) equations in \((N+1)\) unknowns \(c_0, ..., c_{N-1}, c_N\) which can be written in a matrix form

\begin{equation}
Ac^{n+1} = Bc^n \tag{13}
\end{equation}

Where; \(A(c^n), \text{ and } B(c^n)\) are septa-diagonal \((N+1) \times (N+1)\) matrices. Since the matrices \(A\) and \(B\) depend on \(c^n\), the matrix equation \((13)\) is nonlinear. After solving the nonlinear system \((13)\), the boundary parameters \(c_{-3}, c_{-2}, c_{-1}, c_N, c_{N+1}, c_{N+2}, c_{N+3}\) can be computed at each time level from equation \((12)\).

4. THE INITIAL STATE

To determine the initial parameters \(C^0\) from the initial condition on \(U_N(x,0)\), we firstly rewrite equation (4) for the initial condition

\begin{equation}
U_N(x,0) = \sum_{m=3}^{N+3} C_m^0(t) B_m(x) \tag{14}
\end{equation}

Where we seek the values \(C_0\). To do this we require \(U_N(x,0)\) to satisfy the following constraints:
(a) It agrees with the initial condition \( u(x,0) \) at the knots \( x_m, \ m = 0, 1, \ldots, N \).

(b) The first, the second and the third derivatives of the approximate initial condition agree with those of the exact initial condition at both ends of the range. Leading to an equation of the form

\[
AC^n = B
\]

Where the matrix \( A \) is septa-diagonal, i.e. a system of linear equations that can be solved by any of the well-known methods for such cases.

5. STABILITY ANALYSIS

The Von Neumann stability theory is applied and the growth of a Fourier mode

\[
c_j^n = \hat{c}_j^n e^{ikx}
\]

Where \( k \) is the mode number and \( h \) is the element size, and will be determined for the linearised numerical scheme (10). The nonlinear term \( u^2 u_x \) of equation (1) cannot be handled by the Fourier mode method. We assume that the quantity \( u^2 \) in the nonlinear term \( u^2 u_x \) is locally constant and equal to \( C \), so that equation (10) can now be written as (in a similar way to [8]):

\[
\alpha_i c_{j,i+1}^n + \alpha_j c_{j,i}^n + \alpha_j c_{j+1,i}^n + \alpha_j c_{j+1,i}^n + \alpha_j c_{j+1,i}^n = \\
\alpha_j c_{j-1,i}^n + \alpha_j c_{j-1,i}^n + \alpha_j c_{j-1,i}^n + \alpha_j c_{j+1,i}^n + \alpha_j c_{j+1,i}^n + \alpha_j c_{j+1,i}^n
\]

Where \( j = 0, 1, \ldots, N \),

\[
\alpha_1 = 2h^3 - 7\varepsilon k h^2 c - 210 \mu k, \\
\alpha_2 = 240 h^3 - 392 \varepsilon k h^2 c - 1680 \mu k, \\
\alpha_3 = 2382 h^3 - 1715 \varepsilon k h^2 c + 3990 \mu k, \\
\alpha_4 = 4832 h^3, \\
\alpha_5 = 2382 h^3 + 1715 \varepsilon k h^2 c - 3990 \mu k, \\
\alpha_6 = 240 h^3 + 392 \varepsilon k h^2 c + 1680 \mu k, \\
\alpha_7 = 2h^3 + 7\varepsilon k h^2 c + 210 \mu k,
\]

If we insert the Fourier mode (16) in equation (17) we obtain

\[
(a + ib)\hat{c}_{j,i}^{n+1} = (a - ib)\hat{c}_j^n
\]

Where

\[
a = h^3 (2 \cos(3k) + 240 \cos(2k) + 2382 \cos(k) + 4832), \\
b = (7\varepsilon k h^2 c + 210 \mu k) \sin(3k) + (392 \varepsilon k h^2 c + 1680 \mu k) \sin(2k) + (1715 \varepsilon k h^2 c - 3990 \mu k) \sin(k)
\]

We get

\[
\hat{c}_j^{n+1} = g\hat{c}_j^n
\]

Where \( g \) is the growth factor. The growth factor is thus

\[
g = \frac{a - ib}{a + ib}
\]

Then the linearized numerical scheme is unconditionally stable.
6. THE CONSERVATION LAWS

It is of great importance to discuss the conservation laws for our problems, the MKdV equation possesses four polynomial invariants, these invariants can be derived, easily to be shown in that case as follows [1,4]:

\[ C_1 = \int_{-\infty}^{\infty} u \, dx \]
\[ C_2 = \int_{-\infty}^{\infty} u^2 \, dx \]
\[ C_3 = \int_{-\infty}^{\infty} \left( u^4 - \frac{6}{\varepsilon} \mu u_x^2 \right) \, dx \]
\[ C_4 = \int_{-\infty}^{\infty} \left( u^6 - \frac{30}{\varepsilon} \mu u_x^3 + \frac{18}{\varepsilon^2} \mu^2 u_{xx}^2 \right) \, dx \]  

(22)

In our test problems we pay attention to these four invariants, and make sure that these laws are always satisfied.

7. TEST PROBLEM

Following we apply our numerical scheme on the type of nonlinear equation we are handling which is the MKdV equation.

7.1 single soliton

It will known that MKdV equation has the single soliton analytic solution:

\[ u(x,t) = \sqrt{\frac{6c}{\varepsilon}} \sech \left( \sqrt{\frac{c}{\mu}} \left( x - ct - x_0 \right) \right), \]  

(23)

The boundary conditions for the cases are given by an equation (2). To examine the accuracy of the numerical method we have used the L_2 norm to compare the numerical and exact solutions, with \( c = 0.845, \mu = 1, \varepsilon = 3 \) and \( x_0 = 15 \); the range \( 0 \leq x \leq 80 \) is divided into 800 elements of equal length 0.1 our results are given in table (1)

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<tr>
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<th>I_1</th>
<th>I_2</th>
<th>I_3</th>
<th>I_4</th>
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7.2 Two solitons interaction

In this test we choose the initial condition as the sum of two solitary waves of the form

\[ u(x,0) = a_1 \sech \left( \frac{c_1}{\mu} (x - x_1) \right) + a_2 \sech \left( \frac{c_2}{\mu} (x - x_2) \right), \]  

(24)

\[ a_i = \frac{6c_i}{\varepsilon}, i = 1, 2, \]
Where $c_1 = 2$, $c_2 = 1$, $x_1 = 15$, $x_2 = 25$. The conserved quantities are given in Table 2, which almost constant during the interaction simulation.

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### 7.3 Three solitons interaction

In this test we choose the initial condition for three waves

$$u(x,0) = \sum_{i=1}^{3} a_i \text{sech} \left( \frac{c_i}{\mu} (x - x_i) \right), \quad a_i = \sqrt{\frac{6c_i}{\epsilon}}, \quad i = 1, 2, 3$$

where $c_1 = 2$, $c_2 = 1$, $c_3 = 0.5$, $x_1 = 15$, $x_2 = 25$, $x_3 = 35$. The conserved quantities are given in Table 3, which almost constant during the interaction simulation.

<table>
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### CONCLUSION

In this paper, we have solved the MKdV equation using collocation with septic spline. The resulting scheme produced a nonlinear septa-diagonal system. Single soliton, the interaction of two and three solitons are used to assess the performance of the method. The suggested method can be also used in a very efficient way for solving nonlinear PDE’s. Also, numerical results are studied and in each case the four invariants of conservation come out to be almost constant at different time levels and the scheme is unconditionally stable.

### REFERENCES