NEW ALGORITHM FOR SOLVING THE MODIFIED KORTEWEG-DE VRIES (mKdV) EQUATION

K.R. Raslan[1] & H.A. Baghdady[2]
[1]Department of Mathematics, Faculty of Science, AL-Azhar University, Nasr City, Egypt
[2]Department of Mathematics, Faculty of Science, AL-Azhar University,(Girls Branch), Nasr City, Egypt

ABSTRACT
In this paper we will investigate the finite element approach using collocation method with septic spline interpolation functions to solve the Modified Korteweg-de Vries (MKdV) equation. It is unconditionally stable by von Neumann method. The exact soliton solution and the conserved quantities are used to assess the accuracy and to show the robustness of the scheme. The interaction of two solitary waves for different parameters is discussed.

AMS classification: 65N30; 65N35; 65D07; 76B25
Keywords: Finite elements; Collocation; splines; Solitary waves; MKdV

1. INTRODUCTION
In this paper we will introduce a numerical solution for the Modified Korteweg- de Vries equation (MKdV) which is a non-linear partial differential equation which involves both convection and diffusion, in which numerical schemes are the most available method to study its solutions. In this work we introduce the septic spline with finite element method to solve the MKdV equation, and discuss the stability and the accuracy of this solution comparing with the exact solution with some initial and boundary conditions.

2. THE MKdV EQUATION
The MKdV equation in the *x*-direction of the form [1,4,11]

$$u_t + \varepsilon u^2 u_x + \mu u_{xxx} = 0, \quad a \le x \le b \tag{1}$$

Where ε, μ are positive parameters and the subscripts *x* and *t* denote differentiation w.r.t. x and t respectively, with the physical boundary conditions

$$u(a,t) = u(b,t) = 0, u_x(a,t) = u_x(b,t) = 0, u_{xx}(a,t) = u_{xx}(b,t) = 0, \tag{2}$$

in addition to the initial condition $u(x,0) = f(x)$.

3. COLLOCATION WITH SEPTIC SPLINES
Let $a = x_0 \le \ldots \le x_n = b$ be an equipartitioned (n+1) points for [a,b], it is known that the set $B_m(x), m = -3(1)N+3$ form a basis for the space of solution at the collocation points (the knots) [9], be given as

$$B_m(x) = \frac{1}{h^5}\begin{cases} (x - x_{m-4})^7 & , x_{m-4} \le x \le x_{m-3} \\ (x - x_{m-4})^7 - 8(x - x_{m-3})^7 & , x_{m-3} \le x \le x_{m-2} \\ (x - x_{m-4})^7 - 8(x - x_{m-3})^7 + 28(x - x_{m-2})^7 & , x_{m-2} \le x \le x_{m-1} \\ (x - x_{m-4})^7 - 8(x - x_{m-3})^7 + 28(x - x_{m-2})^7 - 56(x - x_{m-1})^7 & , x_{m-1} \le x \le x_m \\ (x_{m+4} - x)^7 - 8(x_{m+3} - x)^7 + 28(x_{m+2} - x)^7 - 56(x_{m+1} - x)^7 & , x_m \le x \le x_{m+1} \\ (x_{m+4} - x)^7 - 8(x_{m+3} - x)^7 + 28(x_{m+2} - x)^7 & , x_{m+1} \le x \le x_{m+2} \\ (x_{m+4} - x)^7 - 8(x_{m+3} - x)^7 & , x_{m+2} \le x \le x_{m+3} \\ (x_{m+4} - x)^7 & , x_{m+3} \le x \le x_{m+4} \\ 0 & , otherwise \end{cases}$$

(3)

Where the values of the septic splines $B_m(x)$, and all its first and second derivatives vanishes outside the interval $[x_{m-4}, x_{m+4}]$.

Our task is to find an approximate solution $u_N(x,t)$ to the solution u(x,t) in the form:

$$U_N(x,t) = \sum_{m=-3}^{N+3} C_m(t) B_m(x) \tag{4}$$

Where $C_m(t)$ are unknown time dependent parameters which are determined from collocation boundary and initial conditions. The nodal values U_m, U'_m, U''_m, U'''_m [2,3,5,6,7] at the notes x_m are obtained from Eqs. (3) and (4) in the following form

$$\begin{aligned}
U_m &= U(x_m) = c_{m-3} + 120c_{m-2} + 1191c_{m-1} + 2416c_m + 1191c_{mi+1} + 120c_{m+2} + c_{m+3} \\
U'_m &= U'(x_m) = \frac{7}{h}(-c_{m-3} - 56c_{m-2} - 245c_{m-1} + 245c_{m+1} + 56c_{m+2} + c_{m+3}) \\
U''_m &= U''(x_m) = \frac{42}{h^2}(c_{m-3} + 24c_{m-2} + 15c_{m-1} - 80c_m + 15c_{m+1} + 24c_{m+2} + c_{m+3}) \\
U'''_m &= U'''(x_m) = \frac{210}{h^3}(-c_{m-3} - 8c_{m-2} + 19c_{m-1} - 19c_{m+1} + 8c_{m+2} + c_{m+3})
\end{aligned} \tag{5}$$

The time derivative of Eq. (1) is discretized by a first order accurate forward difference formula and by using the θ-weighted ($0 \le \theta \le 1$), scheme to the space derivative at two time levels to get the equation

$$\frac{U^{n+1} - U^n}{k} + \varepsilon(\theta(U^2U_x)^{n+1} + (1-\theta)(U^2U_x)^n) + \mu(\theta(U_{xxx})^{n+1} + (1-\theta)(U_{xxx})^n) \tag{6}$$

Where k is the time step and the superscripts n and n+1 are successive time levels. In this work we take θ = 0.5. hence Eq. (6) is written as

$$\frac{U^{n+1} - U^n}{k} + \varepsilon\frac{(U^2U_x)^{n+1} + (U^2U_x)^n}{2} + \mu\frac{(U_{xxx})^{n+1} + (U_{xxx})^n}{2} \tag{7}$$

The nonlinear term in Eq. (7) is approximated by applying Taylor series [10]

$$(U^2U_x)^{n+1} \approx (U^2)^n(U_x)^{n+1} + (U^2)^{n+1}(U_x)^n - (U^2U_x)^n \tag{8}$$

At the nth time step, we denote U, U', U'', U''' at the notes x_m by the following expressions

$$\begin{aligned}
L_{m1} &= c^n_{m-3} + 120c^n_{m-2} + 1191c^n_{m-1} + 2416c^n_m + 1191c^n_{m+1} + 120c^n_{m+2} + c^n_{m+3} \\
L_{m2} &= \frac{7}{h}(-c^n_{m-3} - 56c^n_{m-2} - 245c^n_{m-1} + 245c^n_{m+1} + 56c^n_{m+2} + c^n_{m+3} \\
L_{m3} &= \frac{42}{h^2}(c^n_{m-3} + 24c^n_{m-2} + 15c^n_{m-1} - 80c^n_m + 15c^n_{m+1} + 24c^n_{m+2} + c^n_{m+3}) \\
L_{m4} &= \frac{210}{h^3}(-c^n_{m-3} - 8c^n_{m-2} + 19c^n_{m-1} - 19c^n_{m+1} + 8c^n_{m+2} + c^n_{m+3})
\end{aligned} \tag{9}$$

Using the knots x_m, m = 0, 1, 2,…,N as collocation points, the following recurrence relation at point x_m is obtained using Eqs. (6)-(9)

$$z_1 c_{m-3}^{n+1} + z_2 c_{m-2}^{n+1} + z_3 c_{m-1}^{n+1} + z_4 c_m^{n+1} + z_5 c_{m+1}^{n+1} + z_6 c_{m+2}^{n+1} + z_7 c_{m+3}^{n+1} = 2h^3 (L_{m1} + \frac{\varepsilon k}{2} L_{m1}^2 L_{m2} - \frac{\mu k}{2} L_{m3}) \tag{10}$$

Where

$$\begin{aligned} z_1 &= L_{m0} - 7\varepsilon k h^2 L_{m1}^2 - 210\,\mu k, \\ z_2 &= 120\, L_{m0} - 392\,\varepsilon k h^2 L_{m1}^2 - 1680\,\mu k, \\ z_3 &= 1191\, L_{m0} - 1715\,\varepsilon k h^2 L_{m1}^2 + 3990\,\mu k, \\ z_4 &= 2416\, L_{m0}, \\ z_5 &= 1191\, L_{m0} + 1715\,\varepsilon k h^2 L_{m1}^2 - 3990\,\mu k, \\ z_6 &= 120\, L_{m0} + 392\,\varepsilon k h^2 L_{m1}^2 + 1680\,\mu k, \\ z_7 &= L_{m0} + 7\varepsilon k h^2 L_{m1}^2 + 210\,\mu k, \\ L_{m0} &= 2h^3 (1 + \varepsilon k L_{m1} L_{m2}), \text{ where } m = 0,1,2,...N \end{aligned} \tag{11}$$

The system (10) consists of (N+1) equations in (N+7) unknowns $c_{-3}, c_{-2}, c_{-1}, c_0, ..., c_{N+1}, c_{N+2}, c_{N+3}$. to obtain a unique solution to this system we need six additional constraints. These are obtained from the boundary conditions (2) which require that:

$$\begin{aligned} & c_{-3} + 120\, c_{-2} + 1191\, c_{-1} + 2416\, c_0 + 1191\, c_1 + 120\, c_2 + c_3 = 0, \\ & -7 c_{-3} - 392\, c_{-2} - 1715\, c_{-1} + 1715\, c_1 + 392\, c_2 + 7 c_3 = 0, \\ & 42 c_{-3} + 1008\, c_{-2} + 630\, c_{-1} - 3360\, c_0 + 630\, c_1 + 1008\, c_2 + 42 c_3 = 0, \\ & c_{N-3} + 120\, c_{N-2} + 1191\, c_{N-1} + 2416\, c_N + 1191\, c_{N+1} + 120\, c_{N+2} + c_{N+3} = 0, \\ & -7 c_{N-3} - 392\, c_{N-2} - 1715\, c_{N-1} + 1715\, c_{N+1} + 392\, c_{N+2} + 7 c_{N+3} = 0, \\ & 42 c_{N-3} + 1008\, c_{N-2} + 630\, c_{N-1} - 3360\, c_N + 630\, c_{N+1} + 1008\, c_{N+2} + 42 c_{N+3} = 0 \end{aligned} \tag{12}$$

Eliminating $c_{-3}, c_{-2}, c_{-1}, c_{N+1}, c_{N+2}$ and c_{N+3} from our scheme (10), then (N+1) equations in (N+1) unknowns $c_0, \ldots, c_{N\text{-}1}, c_N$ which can be written in a matrix form

$$Ac^{n+1} = Bc^n \tag{13}$$

Where; $A(c^n)$, and $B(c^n)$ are septa-diagonal $(N+1)\times(N+1)$ matrices. Since the matrices *A* and *B* depend on c^n, the matrix equation (13) is nonlinear. After solving the nonlinear system (13), the boundary parameters $c_{-3}, c_{-2}, c_{-1}, c_{N+1}, c_{N+2}, c_{N+3}$ can be computed at each time level from equation (12).

4. THE INITIAL STATE

To determine the initial parameters C^0 from the initial condition on $U_N(x,0)$, we firstly rewrite equation (4) for the initial condition

$$U_N(x,0) = \sum_{m=-3}^{N+3} C_m^0(t) B_m(x) \tag{14}$$

Where we seek the values C^0. To do this we require $U_N(x,0)$ to satisfy the following constraints:

(a) It agrees with the initial condition $u(x,0)$ at the knots x_m, $m = 0, 1, \ldots, N$.

(b) The first, the second and the third derivatives of the approximate initial condition agree with those of the exact initial condition at both ends of the range. Leading to an equation of the form

$$AC^0 = B \tag{15}$$

Where the matrix *A* is septa-diagonal, i.e. a system of linear equations that can be solved by any of the well-known methods for such cases.

5. STABILITY ANALYSIS

The Von Neumann stability theory is applied and the growth of a Fourier mode

$$c_j^n = \hat{\varepsilon}^n e^{ijkh} \tag{16}$$

Where *k* is the mode number and *h* is the element size, and will be determined for the linearised numerical scheme (10). The nonlinear term $u^2 u_x$ of equation (1) cannot be handled by the Fourier mode method. We assume that the quantity u^2 in the nonlinear term $u^2 u_x$ is locally constant and equal to C, so that equation (10) can now be written as (in a similar way to [8]):

$$\begin{aligned}\alpha_1 c_{j-3}^{n+1} + \alpha_2 c_{j-2}^{n+1} + \alpha_3 c_{j-1}^{n+1} + \alpha_4 c_j^{n+1} + \alpha_5 c_{j+1}^{n+1} + \alpha_6 c_{j+2}^{n+1} + \alpha_7 c_{j+3}^{n+1} = \\ \alpha_7 c_{j-3}^{n} + \alpha_6 c_{j-2}^{n} + \alpha_5 c_{j-1}^{n} + \alpha_4 c_j^{n} + \alpha_3 c_{j+1}^{n} + \alpha_2 c_{j+2}^{n} + \alpha_1 c_{j+3}^{n}\end{aligned} \tag{17}$$

Where j = 0 ,1 ,…, N,

$$\begin{aligned}
\alpha_1 &= 2h^3 - 7\varepsilon k h^2 c - 210\,\mu k, \\
\alpha_2 &= 240\,h^3 - 392\,\varepsilon k h^2 c - 1680\,\mu k, \\
\alpha_3 &= 2382\,h^3 - 1715\,\varepsilon k h^2 c + 3990\,\mu k, \\
\alpha_4 &= 4832\,h^3, \\
\alpha_5 &= 2382\,h^3 + 1715\,\varepsilon k h^2 c - 3990\,\mu k, \\
\alpha_6 &= 240\,h^3 + 392\,\varepsilon k h^2 c + 1680\,\mu k, \\
\alpha_7 &= 2h^3 + 7\varepsilon k h^2 c + 210\,\mu k,
\end{aligned} \tag{18}$$

If we insert the Fourier mode (16) in equation (17) we obtain

$$(a + ib)\hat{\varepsilon}^{n+1} = (a - ib)\hat{\varepsilon}^n \tag{19}$$

Where

$$\begin{aligned}
a &= h^3\left(2\cos(3kh) + 240\cos(2kh) + 2382\cos(kh) + 4832\right), \\
b &= (7\varepsilon k h^2 c + 210\,\mu k)\sin(3kh) + (392\,\varepsilon k h^2 c + 1680\,\mu k)\sin(2kh) \\
&\quad + (1715\,\varepsilon k h^2 c - 3990\,\mu k)\sin(kh)
\end{aligned} \tag{20}$$

We get

$$\hat{\varepsilon}^{n+1} = g\hat{\varepsilon}^n \tag{20}$$

Where *g* is the growth factor. The growth factor is thus

$$g = \frac{a - ib}{a + ib} \tag{21}$$

Then the linearized numerical scheme is unconditionally stable.

6. THE CONSERVATION LAWS

It is of great importance to discuss the conservation laws for our problems, the MKdV equation possesses four polynomial invariants, these invariants can be derived, easily to be shown in that case as follows[1,4]:

$$C_1 = \int_{-\infty}^{\infty} u\,dx$$

$$C_2 = \int_{-\infty}^{\infty} u^2 dx$$

$$C_3 = \int_{-\infty}^{\infty} \left(u^4 - \frac{6}{\varepsilon}\mu u_x^2 \right) dx \qquad (22)$$

$$C_4 = \int_{-\infty}^{\infty} \left(u^6 - \frac{30}{\varepsilon}\mu u^2 u_x^2 + \frac{18}{\varepsilon^2}\mu^2 u_{xx}^2 \right) dx$$

In our test problems we pay attention to these four invariants, and make sure that these laws are always satisfied.

7. TEST PROBLEM

Following we apply our numerical scheme on the type of nonlinear equation we are handling which is the MKdV equation.

7.1 single soliton

It will known that MKdV equation has the single soliton analytic solution:

$$u(x,t) = \sqrt{\frac{6c}{\varepsilon}}\,\mathrm{sech}\left[\sqrt{\frac{c}{\mu}} \left(x - ct - x_0 \right) \right], \qquad (23)$$

The boundary conditions for the cases are given by an equation (2). To examine the accuracy of the numerical method we have used the L_2 norm to compare the numerical and exact solutions, with $c = 0.845, \mu = 1, \varepsilon = 3$ and $x_0 = 15$. the range $0 \le x \le 80$ is divided into 800 elements of equal length 0.1 our results are given in table (1)

Table (1)
Invariants and norms for the single solitary wave
h = 0.1, $0 \le x \le 80$

T	I_1	I_2	I_3	I_4	L_∞	L_2
0.0	4.44288	3.67696	2.07135	1.05018	1.1102E-16	3.055E-16
0.5	4.44288	3.67696	2.07135	1.05018	5.9926E-6	1.3009E-5
1.0	4.44288	3.67696	2.07135	1.05018	1.07322E-5	2.2967E-5
1.5	4.44288	3.67696	2.07135	1.05018	1.68966E-5	3.3504E-5
2.0	4.44288	3.67696	2.07135	1.05018	2.31336E-5	4.4200E-5
2.5	4.44288	3.67695	2.07135	1.05017	2.91204E-5	5.4977E-5
3.0	4.44288	3.67696	2.07135	1.05017	3.5951E-5	6.5730E-5

7.2 Two solitons interaction

In this test we choose the initial condition as the sum of two solitary waves of the form

$$u(x,0) = a_1\,\mathrm{sech}\left(\sqrt{\frac{c_1}{\mu}}(x - x_1) \right) + a_2\,\mathrm{sech}\left(\sqrt{\frac{c_2}{\mu}}(x - x_2) \right),$$
$$a_i = \sqrt{\frac{6c_i}{\varepsilon}}, i = 1,2, \qquad (24)$$

Where $c_1 = 2$, $c_2 = 1$, $x_1 = 15$, $x_2 = 25$. The conserved quantities are given in table 2, which almost constant during the interaction simulation.

Table (2)

T	I_1	I_2	I_3	I_4
0.0	8.88577	9.65938	10.2193	10.6715
0.5	8.88576	9.65937	10.2193	10.6741
1.0	8.88578	9.65936	10.2192	10.6733
1.5	8.88575	9.65935	10.2192	10.6720
2.0	8.88580	9.65933	10.2191	10.6711
2.5	8.88578	9.65932	10.2191	10.6711
3.0	8.88574	9.65930	10.2190	10.6717

7.3 Three solitons interaction
In this test we choose the initial condition for three waves

$$u(x,0)=\sum_{i=1}^{3} a_i \operatorname{sech}\left[\sqrt{\frac{c_i}{\mu}}\left(x-x_i\right)\right], \qquad a_i=\sqrt{\frac{6c_i}{\varepsilon}},\quad i=1,2,3$$

where $c_1=2$, $c_2=1$, $c_3=0.5$, $x_1=15$, $x_2=25$, $x_3=35$. The conserved quantities are given in Table 3, which almost constant during the interaction simulation.

Table (3)

T	I_1	I_2	I_3	I_4
0.0	13.3286	12.5199	11.2285	11.0221
0.5	13.3286	12.5199	11.2285	11.0245
1.0	13.3287	12.5199	11.2284	11.0221
1.5	13.3286	12.5199	11.2284	11.0222
2.0	13.3287	12.5199	11.2283	11.0228
2.5	13.3287	12.5199	11.2282	11.0259
3.0	13.3285	12.5199	11.2281	11.0442

CONCLUSION

In this paper, we have solved the MKdV equation using collocation with septic spline. The resulting scheme produced a nonlinear septa-diagonal system. Single soliton, the interaction of two and three solitons are used to assess the performance of the method. The suggested method can be also used in a very efficient way for solving nonlinear PDE's. Also, numerical results are studied and in each case the four invariants of conservation come out to be almost constant at different time levels and the scheme is unconditionally stable.

REFERENCES

[1]. R. M. Miura, C. S. Gardner, and M. D. Kruskal, Korteweg- de Vries equation and Generalisation.II. Existence of Conservation Laws and Constants of Motion, J. of Math. Phys., 9 (8), 1204-1209,1968.

[2]. J. H. Ahlberg, T. Ito, a Collocation Method For Two-Point Boundary Value Problem, Math. Compt. Vol.29, PP.761-776,1975.

[3]. P. M. Prenter, Splines and Variational Methods, J. Wiley, 1975.

[4]. R. M. Miura, The Korteweg-de Vries equation : A Survey of Results, SIAM Review, 18 (3), 412-459, 1976.

[5]. A. K. Khalifa, Theory and Application of the Collocation method with Splines for Ordinary and Partial Differential Equations, Ph. D, Thesis, Heriot- Watt University, 1979.

[6]. T.J.R. Hughes, Finite Element Methods For Convection Dominated Flows, The American Society of Mechanical Engineers, New York, 1979.

[7]. A. H. A. Ali, Finite Element Solution For The Regularized Long Wave Equation, First ASIAN Computational Fluid Dynamics Conference, 16-19 January, Vol.3, 1201-1204, 1995.

[8]. A. A. Solim and K. R. Raslan, First Integral method for the "improved"KdV Equation , International J. of Nonlinear Science, 8(1),11-18, 2009.

[9]. T.Geyikli, S. B.G. Karakoc, Septic B-Spline Collocation Method for the Numerical Solution of the Modified Equal Width Wave Equation, Applied Mathematics, 2011, 2, 739-749

[10]. Fazal-i-Haq, Inayat Ali shah and Shakeel Ahmad. Septic B-Spline Collocation method for numerical solution of the Equal Width Wave (EW) equation**.** Life Sci. J. 2013;10(1s):253-260] (ISSN: 1097-8135).

[11]. K.R. Raslan, S.H. Youns, H.A. Baghdady Numerical Study for the "Improved" Modefied Korteweg –de Vries Type Equation. IJCSA, 2013, accepted.