ON THE EXTENSION OF WEAK ARMENDARIZ RINGS RELATIVE TO A MONOID

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ABSTRACT

For a monoid $M$, we introduce the concept of weak $3-M$-Armendariz rings, which are a common generalization of $3-M$-Armendariz rings and weak $M$-Armendariz rings, and investigates its properties. Moreover, this paper proves that a ring $R$ is a weak $3-M$-Armendariz if and only if for any $n$, the $n$-by-$n$ upper triangular matrix ring $T_n(R)$ over $R$ is a weak $3-M$-Armendariz. If the ideal $I$ is a reduced and $R/I$ is a weak $3-M$-Armendariz, then $R$ is a weak $3-M$-Armendariz, where $M$ is strictly totally ordered monoid. Also we show that if a ring $R$ satisfy condition (P) and a weak $3-M$-Armendariz, then $R$ is a weak $(M\times N)$-Armendariz, where $N$ is a unique product monoid.

Keywords: unique product monoid, 3-Armendariz ring, 3-M-Armendariz ring, weak 3-M-Armendariz ring.

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1. INTRODUCTION

Throughout this paper, $R$ and $M$ denote an associative ring, not necessarily with identity and a monoid, respectively. Given a ring $R$, the polynomial ring over $R$ is denoted by $R[x]$. We denote by $T_n(R)$ the $n$-by-$n$ upper triangular matrix ring over $R$. The study of Armendariz rings was initiated by Armendariz [5] and Rege and Chhawchharia [9]. A ring $R$ is called Armendariz if whenever polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$, for each $i, j$. (The converse is always true.) Some properties of Armendariz rings have been studied in Rege and Chhawchharia [9], Anderson and Camillo [3], Kim and Lee [10], Hong et al. [2], and Lee and Wong [12]. Suiyi [15] introduced the notion of 3-Armendariz rings. A ring $R$ is called a 3-Armendariz ring if whenever polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m$, $h(x) = c_0 + c_1 x + \cdots + c_k x^k \in R[x]$, satisfy $f(x)g(x)h(x) = 0$, then $a_i b_j c_k = 0$, for each $i, j, k$. Zhongkui [7], studied a generalization of Armendariz rings, which are called $M$-Armendariz rings, where $M$ is a monoid. A ring $R$ is called $M$-Armendariz if whenever elements $\alpha = a_1 g_1 + \cdots + a_n g_n$, $\beta = b_1 h_1 + \cdots + b_m h_m \in R[M]$, satisfy $\alpha \beta = 0$, then $a_i b_j = 0$ for each $i, j$, where $g_i, h_j \in M$. A ring $R$ is called reduced if it has no nonzero nilpotent elements. Reduced rings are Armendariz by Armendariz [5, Lemma 1.1] and subrings of an Armendariz ring are also Armendariz. A ring $R$ is called abelian if every idempotent is central. Armendariz ring are abelian by Kim and Lee [10]. Subrings of $M$-Armendariz ring are also $M$-Armendariz by Zhongkui [7]. Subrings of a 3-Armendariz ring are also 3-Armendariz by Suiyi [15]. Liu and Zhao [8]. Introduced the notion of weak Armendariz. A ring $R$ is called weak Armendariz if whenever polynomials $f(x) = a_0 + \cdots + a_n x^n$, $g(x) = b_0 + \cdots + b_m x^m \in R[x]$ satisfy $f(x)g(x)h(x) = 0$, then $a_i b_j c_k \in \text{nil}(R)$ for each $i, j, k$. Elshokry and et al. [1]. Introduced the notion of weak 3-Armendariz, where
\( M \) is the a monoid. A ring \( R \) is called 3-\( M \)-Armendariz if whenever elements
\[ \alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m, \gamma = c_1l_1 + \cdots + c_ml_m \in R[M], \]
satisfy \( \alpha\beta\gamma = 0 \), then \( a_ib_jc_k = 0 \) for each \( i, j, k \) where \( g_i, h_j, l_k \in M \). In [16], a ring \( R \) is called weak \( M \)-Armendariz if whenever elements
\[ \alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m \in R[M], \]
satisfy \( \alpha\beta = 0 \), then \( a_ib_j \in \text{nil}(R) \) for each \( i, j \). Clearly, \( M \)-Armendariz rings are weak \( M \)-Armendariz.

Recall that a monoid \( M \) is called a u.p.-monoid (unique product monoid) if for any two non-empty finite subsets \( A, B \subseteq M \) there exists an element \( g \in M \) uniquely presented in the form \( ab \) where \( a \in A \) and \( b \in B \). The class of u.p.-monoids is quite large and important (see Birkenmeier and Park [6], Passman [4]). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid \( M \) has non unity element of finite order.

Motivated by results in Elshokry and et al. [1], Suiyi [15], Zhongkui [7], Rege and Chhawchharia [9], Zhongkui and Zhao [8], Kim and Lee [10], Wu Hui-feng [14] and Cuiping and Jianlong [16], we will investigate a common generalization of \( M \)-Armendariz and 3-\( M \)-Armendariz rings, which we called weak 3-\( M \)-Armendariz rings.

2. **Weak 3-Armendariz rings relative to a monoid**

For a monoid \( M \), \( e \) will always stand for the identity of \( M \). If \( R \) is a ring, then \( R[M] \) denotes the monoid ring over \( R \).

**Definition 2.1** Let \( M \) be a monoid. A ring \( R \) is called a weak 3-\( M \)-Armendariz ring if whenever elements
\[ \alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m \text{ and } \gamma = c_1l_1 + \cdots + c_ml_m \in R[M], \]
satisfy \( \alpha\beta\gamma = 0 \), then \( a_ib_jc_k \in \text{nil}(R) \) for each \( i, j \) and \( k \), where \( a_i, b_j, c_k \in R \) and \( g_i, h_j, l_k \in M \).

**Proposition 2.2** Every subring of a weak 3-\( M \)-Armendariz rings is a weak 3-\( M \)-Armendariz.

**Proof.** It is obvious.

We introduce the following notation (see [15]).

**Condition (P):** For all \( a, b, c \in R \), if \( (abc)^2 = 0 \), then \( abc = 0 \).

**Lemma 2.3** [13, Proposition 1]. If \( R \) is a reduced ring, then \( R \) satisfies the condition (P), but the converse is not true.

**Lemma 2.4** [6, Lemma 1.1]. Assume \( M \) is a u.p.-monoid. Then \( M \) is cancellative (i.e., for \( g, h, x \in M \), if \( gx = hx \) or \( xg = xh \), then \( g = h \)).

**Theorem 2.5** Let \( M \) be a u.p.-monoid and \( \text{nil}(R) \) an ideal of \( R \). Then \( R \) is weak 3-\( M \)-Armendariz.

**Proof.** Let \( \alpha = \sum_{i=1}^{n} a_ig_i, \beta = \sum_{j=1}^{m} b_jh_j \text{ and } \gamma = \sum_{k=1}^{l} c_kl_k \in R[M], \) satisfy \( \alpha\beta\gamma = 0 \). Since \( \text{nil}(R) \) is an ideal of \( R \), the ring \( \overline{R} = R/\text{nil}(R) \) is reduced. By Lemma 2.3, \( \overline{R} = R/\text{nil}(R) \) satisfies the condition (P) and so 3-\( M \)-Armendariz, by [1, Theorem 2.6]. Also, \( \alpha\beta\gamma = 0 \) implies that \( \overline{\alpha\beta\gamma} = \overline{0} \). So \( \overline{a_i} \cdot \overline{b_j} \overline{c_k} = \overline{0} \), for each \( i, j \) and \( k \), since \( \overline{R} \) is 3-\( M \)-Armendariz. Thus \( a_ib_jc_k \in \text{nil}(R) \), for each \( i, j \) and \( k \), and the result follows.

**Proposition 2.6** Let \( M \) be a u.p.-monoid and \( R \) a reduced ring. Then \( R \) is weak 3-\( M \)-Armendariz.

**Proof.** Since \( R \) is reduced, hence \( \text{nil}(R) = 0 \) is an ideal of \( R \). Thus, the result follows from Theorem 2.5.
Corollary 2.7 Let $M$ be a u.p.-monoid and $R$ a ring satisfying the condition $(P)$. Then $R$ is a weak $3-M$-Armendariz.

Proof. Let $M$ be a u.p.-monoid and $R$ a ring satisfying the condition $(P)$. Then by [1, Theorem 2.6], $R$ is 3-M-Armendariz ring. Thus, $R$ is a weak 3-M-Armendariz.

Let $(M, \leq)$ be an ordered monoid. If for any $g, g', h \in M, g < g'$ implies that $gh < g'h$ and $hg < hg'$, then $(M, \leq)$ is called a strictly ordered monoid.

Corollary 2.8 Let $M$ be a strictly totally ordered monoid and $R$ a ring satisfying the condition $(P)$. Then $R$ is a weak 3-M-Armendariz.

Corollary 2.9 If a ring $R$ satisfies condition $(P)$, then $R$ is a weak 3-$\mathcal{Z}$-Armendariz that is, for any
\[ \alpha = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0 x^0, \beta = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0 x^0 \]
and
\[ \gamma = c_r x^r + c_{r-1} x^{r-1} + \cdots + c_1 x + c_0 \in R[x, x^{-1}], \]
if $\alpha \beta \gamma = 0$, then $a b c_k \in \text{nil}(R)$ for $-m \leq i \leq p, -n \leq j \leq q$ and $-t \leq k \leq s$.

In [1, Theorem 2.9], it was shown that if $I$ is a reduced ideal of $R$ such that $R/I$ is 3-M-Armendariz, then $R$ is 3-M-Armendariz. Here we have the following result for a weak 3-M-Armendariz property.

Theorem 2.10 Let $M$ be a strictly totally ordered monoid and $I$ an ideal of $R$. If $I$ is reduced and $R/I$ is a weak 3-M-Armendariz, then $R$ is a weak 3-M-Armendariz.

Proof. Let $\alpha, \beta, \gamma \in R[M]$ be such that $\alpha \beta \gamma = 0$. We write $\alpha = a_1 g_1 + \cdots + a_n g_n, \beta = b_1 h_1 + \cdots + b_m h_m$ and $\gamma = c_1 l_1 + \cdots + c_r l_r \in R[M]$, with
\[ g_1 < g_2 < \cdots < g_n, h_1 < h_2 < \cdots < h_m, l_1 < l_2 < \cdots < l_r. \]
We will use transfinite induction on the strictly totally ordered set $(M, \leq)$ to show that $a b c_k \in \text{nil}(R)$, for any $i, j$ and $k$. Note that in
\[ (R/I)[M], (\tilde{a}_1 g_1 + \tilde{a}_2 g_2 + \cdots + \tilde{a}_n g_n)(\tilde{b}_1 h_1 + \tilde{b}_2 h_2 + \cdots + \tilde{b}_m h_m)(\tilde{c}_1 l_1 + \tilde{c}_2 l_2 + \cdots + \tilde{c}_r l_r) = 0. \]
Thus we have $a b c_k \in I$ for all $i, j$ and $k$ with $1 \leq i \leq n, 1 \leq j \leq m$ and $1 \leq k \leq r$, since $R/I$ is a weak 3-M-Armendariz.

If there exist $1 \leq i \leq n, 1 \leq j \leq m$ and $1 \leq k \leq r$, such that $g_i h_j l_k = g_i h_k l_j$, then $g_i \leq g_j, h_i \leq h_j$ and $l_i \leq l_k$. If $g_i < g_j$, then $g_i h_j l_k = g_i h_k l_j < g_i h_j l_k$ by assumption, a contradiction, hence $g_i = g_j$. Similarly, $h_i = h_j$ and $l_i = l_k$. Hence $a_i b_j c_k = 0 \in \text{nil}(R)$. Now suppose that $w \in M$ is such that for any $g_i, h_j$ and $l_k$, if $g_i h_j l_k < w$, then $a_i b_j c_k = 0$. We will show that $a_i b_j c_k \in \text{nil}(R)$, for any $g_i, h_j$ and $l_k$, with $g_i h_j l_k = w$.

Set $X = \{(g_i, h_j, l_k) \mid g_i h_j l_k = w\}$. Then $X$ is a finite set. We write $X$ as $\{(g_i, h_j, l_k) \mid t = 1, 2, \cdots, u\}$ such that
\[ g_{i_1} < g_{i_2} < \cdots < g_{i_u}. \]
We claim that
\[ h_{j_1} l_{k_1} < h_{j_2} l_{k_2} < h_{j_3} l_{k_3}. \]
In fact, if $h_{j_1} l_{k_1} < h_{j_2} l_{k_2}$, then
\[ w = g_{i_1} h_{j_1} l_{k_1} < g_{i_2} h_{j_2} l_{k_2} < g_{i_3} h_{j_3} l_{k_3} = w. \]

a contradiction. If \( h_1 l_1 = h_2 l_2 \), then from \( g_1 h_1 l_1 = w = g_1 h_2 l_2 \) it follows that \( g_1 = g_2 \), a contradiction again. Thus, \( h_2 l_2 < h_1 l_1 \). Similarly we have the claim. For any \( t \geq 2 \), \( g_1 h_1 l_t < g_1 h_1 l_{t-1} = w \), and thus, by induction hypothesis, we have \( a_1 b_j c_k \in \text{nil}(R) \). Then we have \( b_j c_k Ia_1 = 0 \), since \( b_j c_k \subseteq I \). Thus for any \( t \geq 2 \),

\[
(a_1 b_j c_k)(a_1 b_j c_k) = (a_1 b_j c_k)(a_1 b_j c_k) = (a_1 b_j c_k)(a_1 b_j c_k) I(1) = a_1 (b_j c_k I) b_j c_k = 0.
\]

Which implies that \( (a_1 b_j c_k)(a_1 b_j c_k)^2 = 0 \). Now, from

\[
\sum_{(g_1, h_1, l_1) \in X} (a_1 b_j c_k) = \sum_{i=1}^{u} a_1 b_j c_k = 0,
\]

it follows that

\[
(\sum_{i=1}^{u} a_1 b_j c_k)(a_1 b_j c_k)^2 = (a_1 b_j c_k)^3 = 0.
\]

Since \( a_1 b_j c_k \in I \) and \( I \) is reduced, we have \( a_1 b_j c_k \in \text{nil}(R) \). Thus, \( \sum_{i=1}^{u} a_1 b_j c_k = 0 \). Multiplying \( (a_1 b_j c_k)^2 \) on \( \sum_{i=1}^{u} a_1 b_j c_k = 0 \), from the right-hand side, we obtain \( a_1 b_j c_k \in \text{nil}(R) \), by the same way as the above. Continuing this process, we can prove \( a_1 b_j c_k \in \text{nil}(R) \), for \( t = 1, 2, \ldots, u \). Thus, \( a_1 b_j c_k \in \text{nil}(R) \) for any \( i, j \) and \( k \) with \( g_1 h_1 l_{t-1} = w \). Therefore, by transfinite induction, \( a_1 b_j c_k \in \text{nil}(R) \) for any \( i, j \) and \( k \). Thus \( R \) is a weak 3-M-Armendariz.

Recall that a monoid \( M \) is called torsion-free if the following property holds: if \( g_1 h \in M \) and \( k \geq 1 \) are such that \( g^k = h^k \), then \( g = h \).

**Corollary 2.11** Let \( M \) be a commutative, cancellative, and torsion-free monoid. If one of the following conditions holds, then \( R \) is a weak 3-M-Armendariz.

1. \( R \) satisfies condition \((P)\).
2. \( R/I \) is a weak 3-M-Armendariz for some ideal \( I \) of \( R \), and \( I \) is reduced.

**Proof.** If \( M \) is commutative, cancellative, and torsion-free, then, by Ribenboim [11], there exists a compatible strict total order \( \leq \) on \( M \). Now the results follow from Corollary 2.8 and Theorem 2.10.

**Proposition 2.12** Suppose that \( R \) is a weak 3-M-Armendariz, \( n \geq 3 \). If \( \alpha_1, \alpha_2, \ldots, \alpha_n \in R[M] \) are such that \( \alpha_1 \alpha_2 \cdots \alpha_n = 0 \), then \( a_1 a_2 \cdots a_n \in \text{nil}(R) \), where \( a_i \) is a coefficient of \( \alpha_i \).

**Proof.** It follows easily from the definition.

In [16], Cuiping and Jianlong showed that a ring \( R \) is weak \( M \)-Armendariz if and only if \( T_n(R) \) is weak \( M \)-Armendariz. For weak 3-M-Armendariz, in the following we will give more results:

**Theorem 2.13** Let \( M \) be a monoid with \( |M| \geq 2 \). Then \( R \) is a weak 3-M-Armendariz if and only if, for any \( n \), \( T_n(R) \) is a weak 3-M-Armendariz.
Proof. We note that any subring of weak $3M$-Armendariz rings is weak $3M$-Armendariz. Thus if $T_n(R)$ is a weak $3M$-Armendariz ring, then $R$ is a weak $3M$-Armendariz ring. Conversely, Let $\alpha, \beta, \gamma \in R[M]$, we write $\alpha = A_1g_1 + \cdots + A_ng_n, \beta = B_1h_1 + \cdots + B_1h_m$ and $\gamma = C_1l_1 + \cdots + C_1l_r \in R[M]$ be elements of $T_n(R)[M]$. Assume that $\alpha \beta \gamma = 0$. It is easy to see that there exists an isomorphism of rings $T_n(R)[M] \rightarrow T_n(R[M])$ define by:

$$
\begin{align*}
\sum_{i=1}^{p} a_i^j g_i &\mapsto \left( a_{11}^i & a_{12}^i & a_{13}^i & \cdots & a_{1n}^i \\
0 & a_{22}^i & a_{23}^i & \cdots & a_{2n}^i \\
0 & 0 & a_{33}^i & \cdots & a_{3n}^i \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{nn}^i
\right)
\end{align*}
$$

Assume that

$$
A_i = \begin{pmatrix}
a_{11}^i & a_{12}^i & a_{13}^i & \cdots & a_{1n}^i \\
0 & a_{22}^i & a_{23}^i & \cdots & a_{2n}^i \\
0 & 0 & a_{33}^i & \cdots & a_{3n}^i \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{nn}^i
\end{pmatrix}, B_j = \begin{pmatrix}
b_{11}^j & b_{12}^j & b_{13}^j & \cdots & b_{1n}^j \\
0 & b_{22}^j & b_{23}^j & \cdots & b_{2n}^j \\
0 & 0 & b_{33}^j & \cdots & b_{3n}^j \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & b_{nn}^j
\end{pmatrix}
$$

and

$$
C_k = \begin{pmatrix}
c_{11}^k & c_{12}^k & c_{13}^k & \cdots & c_{1n}^k \\
0 & c_{22}^k & c_{23}^k & \cdots & c_{2n}^k \\
0 & 0 & c_{33}^k & \cdots & c_{3n}^k \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & c_{nn}^k
\end{pmatrix}
$$
Then

\[
\begin{pmatrix}
\sum_{i=1}^{p} a_{1i} g_i & \sum_{i=1}^{p} a_{12} g_i & \sum_{i=1}^{p} a_{13} g_i & \cdots & \sum_{i=1}^{p} a_{1n} g_i \\
0 & \sum_{i=1}^{p} a_{22} g_i & \sum_{i=1}^{p} a_{23} g_i & \cdots & \sum_{i=1}^{p} a_{2n} g_i \\
0 & 0 & \sum_{i=1}^{p} a_{33} g_i & \cdots & \sum_{i=1}^{p} a_{3n} g_i \\
0 & 0 & 0 & \cdots & \sum_{i=1}^{p} a_{nn} g_i \\
\end{pmatrix} \times
\begin{pmatrix}
\sum_{j=1}^{q} b_{1j} h_j & \sum_{j=1}^{q} b_{12} h_j & \sum_{j=1}^{q} b_{13} h_j & \cdots & \sum_{j=1}^{q} b_{1n} h_j \\
0 & \sum_{j=1}^{q} b_{22} h_j & \sum_{j=1}^{q} b_{23} h_j & \cdots & \sum_{j=1}^{q} b_{2n} h_j \\
0 & 0 & \sum_{j=1}^{q} b_{33} h_j & \cdots & \sum_{j=1}^{q} b_{3n} h_j \\
0 & 0 & 0 & \cdots & \sum_{j=1}^{q} b_{nn} h_j \\
\end{pmatrix} = 0.
\]

It follows that

\[
(\sum_{i=1}^{p} a_{ss} g_i, \sum_{j=1}^{q} b_{ss} h_j, \sum_{k=1}^{d} c_{ss} l_k) = 0, \quad s = 1, 2, \ldots, n
\]

Since \( R \) is weak 3-\( M \)-Armendariz, there exists \( m_{ijk} \in \mathbb{N} \) such that \( (a_{ss} b_{ss} c_{ss})^{m_{ijk}} = 0 \) for any \( s \) and any \( i, j, k \). Let \( m_{ijk} = \max\{m_{ijk1}, m_{ijk2}, \ldots, m_{ijkn}\} \). Then
Thus \((A_iB_jC_k)^{m_{ijk}} = n\) and so \(A_iB_jC_k \in \text{nil}(T_n(R))\). This shows that \(T_n(R)\) is a weak 3-M-Armendariz ring.

**Corollary 2.14** Let \(M\) be a monoid. If a ring \(R\) is 3-M-Armendariz ring, then, for any \(n\), \(T_n(R)\) is a weak 3-M-Armendariz ring.

**Proposition 2.15** Let \(M\) be a monoid. A ring \(R\) is a weak 3-M-Armendariz if and only if the trivial extension \(T(R, R)\) is a weak 3-M-Armendariz ring.

**Proof.** It follows from Theorem 2.13.
From Theorem 2.13, one may suspect that if \(R\) is a weak 3-M-Armendariz then every n-by-n full matrix ring \(M_n(R)\) over \(R\) is a weak 3-M-Armendariz, where \(n \geq 2\). But the following example erases the possibility.

**Example 2.16** Let \(R\) be a ring and let \(S = M_2(F)\). Let

\[
\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g
\]

and

\[
\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} g
\]

be elements in \(S[M]\), where \(e \neq g \in M\). Then \(\alpha\beta\gamma = 0\). But

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}
\]

is not nilpotent. Thus \(S\) is not weak 3-M-Armendariz.

Clearly, every 3-M-Armendariz ring is a weak 3-M-Armendariz, but the converse is not true by the following example.

**Example 2.17** Let \(R\) be a weak 3-M-Armendariz ring. Then the ring
is not $3 \cdot M$-Armendariz by [1, Example 2.14], when $n \geq 4$, but $R_n$ is a weak $3 \cdot M$-Armendariz ring by Theorem 2.13, since any subring of a weak $3 \cdot M$-Armendariz rings is a weak $3 \cdot M$-Armendariz.

**Proposition 2.18** The class of weak $3 \cdot M$-Armendariz rings is closed under finite direct products.

**Proof.** Let $R = \prod_{i=1}^{p} R_s$ be the finite direct product of $R_s$ where $\rho = \{1, 2, \ldots, p\}$, $R_s$ is weak $3 \cdot M$-Armendariz ring. Suppose $\alpha \beta \gamma = 0$ for some elements $\alpha = a_1 a_1 + \cdots + a_n a_n$, $\beta = b_1 b_1 + \cdots + b_m b_m$ and $\gamma = c_1 c_1 + \cdots + c_l c_l \in R[M]$, where $a_i = (a_{i_1}, a_{i_2}, \ldots, a_{i_p}), b_j = (b_{j_1}, b_{j_2}, \ldots, b_{j_p}), c_k = (c_{k_1}, c_{k_2}, \ldots, c_{k_p})$, are elements of the product ring $R$. Set $\alpha_s = \sum_{i=1}^{n} a_i a_i, \beta_s = \sum_{j=0}^{m} b_j b_j$ and $\gamma_s = \sum_{k=0}^{r} c_k c_k \in R[M]$. Since $\alpha \beta \gamma = 0$ then $\alpha_s \beta_s \gamma_s = 0 = \sum_{i+j+k=r} a_i a_j a_k$, $0 \leq i + j + k \leq p$, and so $\sum_{i+j+k=r} (a_i b_j c_k) = 0$, $1 \leq s \leq p$. Thus $\alpha_s \beta_s \gamma_s = 0$ in $R_i [M], 1 \leq s \leq p$. Since $R_s$ is weak $3 \cdot M$-Armendariz rings, we have $a_i b_j c_k \in \text{nil}(R_s)$. Now, for each $i, j, k$, there exist positive integers $m_{ijk}$ such that $(a_i b_j c_k)^{m_{ijk}} = 0$, in the ring $R_s, 1 \leq s \leq p$. If we take $m_{ijk} = \max \{m_{ijk} : 1 \leq s \leq p\}$, then it is clear that $(a_i b_j c_k)^{m_{ijk}} = 0$. Therefore $a_i b_j c_k \in \text{nil}(R)$. This means that $R$ is a weak $3 \cdot M$-Armendariz.

Recall that an element $u$ of a ring $R$ is right regular if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, left regular elements can be defined. An element is regular if it is both left and right regular (and hence not a zero divisor).

**Theorem 2.19** Let $R$ be a ring and $\Delta$ be a multiplicative monoid in $R$ consisting of central regular elements. Then $R$ is a weak $3 \cdot M$-Armendariz if and only if $\Delta^{-1} R$ is also weak $3 \cdot M$-Armendariz.

**Proof.** Let $R$ be a weak $3 \cdot M$-Armendariz ring, and $S = \Delta^{-1} R$. Put $\alpha \beta \gamma = 0$, where $\alpha = \sum_{i=1}^{n} a_i a_i, \beta = \sum_{j=0}^{m} b_j b_j$ and $\gamma = \sum_{k=0}^{r} c_k c_k \in S[M]$. We may assume that $a_i = \varepsilon_i u^{-1}, b_j = \eta_j v^{-1}$ and $c_k = \mu_k w^{-1}$ with $\varepsilon_i, \eta_j, \mu_k$ are in $R$ for all $i, j$ and $k$, and $u, v, w \in \Delta$. We will show that $a_i b_j c_k \in \text{nil}(S[M])$. Now we have
\[ 0 = \alpha \beta \gamma \]
\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{r} a_i b_j c_k e_i h_j l_k \]
\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{r} \epsilon_i \eta_j \mu_k u^{-1} v^{-1} w^{-1} e_i h_j l_k \]
\[ = (\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{r} \epsilon_i \eta_j \mu_k e_i h_j l_k) (uvw)^{-1}. \]

Hence
\[ \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{r} \epsilon_i \eta_j \mu_k e_i h_j l_k = 0 \]
in \( R[M] \). Since \( R \) is a weak \( 3-M \)-Armendariz, \( \epsilon_i \eta_j \mu_k \in \text{nil}(R) \), for all \( i, j \) and \( k \) and so \( a_i b_j c_k = \epsilon_i u^{-1} \eta_j v^{-1} \mu_k w^{-1} = \epsilon_i \eta_j \mu_k (uvw)^{-1} \in \text{nil}(S[M]) \), for all \( i, j, k \). Thus, \( S \) is a weak \( 3-M \)-Armendariz. The converse follows from Proposition 2.2.

**Proposition 2.20** Let \( M \) be a monoid, and \( R \) be a ring and \( e \) an idempotent of \( R \). If \( e \) is central in \( R \), then the following statements are equivalent:

1. \( eR \) is a weak \( 3-M \)-Armendariz;
2. \( (1-e)R \) are weak \( 3-M \)-Armendariz.

**Proof.** (1) \( \Rightarrow \) (2) is obvious since \( eR \) and \( (1-e)R \) are subrings of \( R \). (2) \( \Rightarrow \) (1) Let \( \alpha = \sum_{i=1}^{n} a_i g_i \), \( \beta = \sum_{j=1}^{m} b_j h_j \) and \( \gamma = \sum_{k=1}^{r} c_k l_k \in R[M] \) be such that \( \alpha \beta \gamma = 0 \). Let \( \alpha_1 = e \alpha, \alpha_2 = (1-e) \alpha, \beta_1 = e \beta, \beta_2 = (1-e) \beta \) and \( \gamma_1 = e \gamma, \gamma_2 = (1-e) \gamma \). Then \( \alpha_1 \beta_1 \gamma_1 = 0 \) and \( \alpha_2 \beta_2 \gamma_2 = 0 \). Since \( eR \) and \( (1-e)R \) are weak \( 3-M \)-Armendariz, there exist \( m_{ij} \) and \( n_{ij} \) such that
\[ (ea_i b_j c_k)^{m_{ij}} = (e(a_i b_j c_k))^{m_{ij}} \in \text{nil}(eR) \]
and
\[ ((1-e)a_i (1-e)b_j (1-e)c_k)^{n_{ij}} = ((1-e)(a_i b_j c_k))^{n_{ij}} \in \text{nil}((1-e)R). \]
Thus, \( e(a_i b_j c_k)^{m_{ij}} = 0 \) and \( (1-e)(a_i b_j c_k)^{n_{ij}} = 0 \). Let \( s_{ijk} = \max\{m_{ijk}, n_{ijk}\} \). Then \( e(a_i b_j c_k)^{s_{ijk}} = 0 \) and \( (1-e)(a_i b_j c_k)^{s_{ijk}} = 0 \).

Thus, \( (a_i b_j c_k)^{s_{ijk}} \in \text{nil}(R) \). This means that \( R \) is weak \( 3-M \)-Armendariz.

**Proposition 2.21** Let \( M \) be a monoid, and \( R \) be a ring and \( I \) an ideal of \( R \) such that \( R/I \) is a weak \( 3-M \)-Armendariz. If \( I \subseteq \text{nil}(R) \), then \( R \) is weak \( 3-M \)-Armendariz.

**Proof.** Let \( \alpha = \sum_{i=1}^{n} a_i g_i, \beta = \sum_{j=1}^{m} b_j h_j \) and \( \gamma = \sum_{k=1}^{r} c_k l_k \in R[M] \) be such that \( \alpha \beta \gamma = 0 \). Then
\[ (\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{m} b_j h_j)(\sum_{k=1}^{r} c_k l_k) = 0. \]
Since \( R/I \) is a weak \( 3-M \)-Armendariz, we have that \( (\bar{a_i} \bar{b_j} \bar{c_k}) \in \text{nil}(R/I) \). Hence \( a_i b_j c_k \in I \). Since \( I \subseteq \text{nil}(R) \) then \( a_i b_j c_k \in \text{nil}(R) \). This means that \( R \) is a weak \( 3-M \)-Armendariz.

In [7, proposition 2.1], it was shown that if \( R \) is a reduced and \( M \)-Armendariz ring, then \( R[M] \) is \( N \)-Armendariz, where \( M \) is a monoid and \( N \) a \( u.p.-\)monoid. Elshokry and et al. [1, proposition 3.1], it was shown that if \( R \) satisfies the condition \( (P) \) and \( 3-M \)-Armendariz, then \( R[M] \) is \( 3-N \)-Armendariz, where \( N \) is a
Proposition 2.22 Let M be a monoid and N a u.p. -monoid. If R is a semicommutative ring which is also weak 3-M -Armendariz, then R[M] is a weak 3-N -Armendariz.

Proof. Suppose that $\alpha = a_1g_1 + \cdots + a_ng_n$, $\beta = b_1h_1 + \cdots + b_mh_m$ and $\gamma = c_1l_1 + \cdots + c_ml_m \in R[M]$, such that $(\alpha\beta\gamma)^2 = 0$. Then $(a_ib_jc_k)^2 = 0$ for all $i$, $j$, and $k$, since $R$ satisfies condition $(P)$ and a weak 3-M -Armendariz. Thus, $a_ib_jc_k \in \text{nil}(R)$ for all $i$, $j$, and $k$. Hence $\alpha\beta\gamma = 0$. This shows that $R[M]$ satisfies condition $(P)$. Now the result follows from Theorem 2.5.

Lemma 2.23 [16, Lemma 3] Let R be a semicommutative ring and M a monoid. If $a_1, \cdots, a_n \in \text{nil}(R)$, then $a_1g_1 + \cdots + a_ng_n \in \text{nil}(R[M]).$

Proposition 2.24 Let M be a monoid and N a u.p. -monoid. If R a semicommutative ring which is also weak 3-M -Armendariz, then $R[N]$ is a weak 3-M -Armendariz.

Proof. It is easy to see that there exists an isomorphism of rings $R[N][M] \cong R[M][N]$ defined by

$$
\sum_p \left( \sum_i a_{ip}n_i \right)m_p \rightarrow \sum_i \left( \sum_p a_{ip}m_p \right)n_i.
$$

Now suppose that $\alpha_i, \beta_j, \gamma_k \in R[N]$ are such that $(\sum_i \alpha_i g_i)(\sum_j \beta_j h_j)(\sum_k \gamma_k l_k) = 0$, where $g_i, h_j, l_k \in M$.

We will show that $\alpha_i\beta_j\gamma_k \in \text{nil}(R[N])$ for all $i$, $j$, and $k$. Assume that $\alpha_i = \sum_p a_{ip}n_p, \beta_j = \sum_q b_{jq}n_q'$ and $\gamma_k = \sum_s c_{ks}n''_s$, where $n_p, n_q', n''_s \in N$ for all $p, q$, and $s$. Then

$$
(\sum_i a_{ip}n_p g_i)(\sum_j b_{jq}n_q' h_j)(\sum_k c_{ks}n''_s l_k) = 0.
$$

Thus, in $R[M][N]$ we have

$$
(\sum_p \left( \sum_i a_{ip}g_i \right)n_p)(\sum_q \left( \sum_j b_{jq}h_j \right)n_q')(\sum_s \left( \sum_k c_{ks}l_k \right)n''_s) = 0.
$$

By Proposition 2.22, $R[M]$ is a weak 3-N -Armendariz. $(\sum a_{ip}g_i)(\sum b_{jq}h_j)(\sum c_{ks}l_k) = 0$ for all $p, q$, and $s$. Since $R$ is a weak 3-M -Armendariz, $a_{ip}b_{jq}c_{ks} \in \text{nil}(R)$ for all $i, j, k, p, q, s$. Hence $\alpha_i\beta_j\gamma_k = 0$. By Lemma 2.23, this means that $R[N]$ is a weak 3-M -Armendariz.

Theorem 2.25 Let M be a monoid and N a u.p. -monoid. If R a semicommutative ring which is also a weak 3-M -Armendariz, then $R$ is a weak 3-$(M \times N)$ -Armendariz.

Proof. Suppose that $\sum_{i=1}^t a_i(m_i, n_i)$ is in $R[M \times N]$. Without loss of generality, we assume that 

$$
\{n_1, n_2, \ldots, n_t\} = \{n_1, n_2, \ldots, n_t\}, \text{ with } n_i \neq n_j \text{ when } 1 \leq i \neq j \leq t.
$$

For any $1 \leq p \leq t$, denote $A_p = \{i \mid 1 \leq i \leq s, n_i = n_p\}$. Then $\sum_{p=1}^t \sum_{i \in A_p} (a_i m_i) n_p \in R[M][N]$. Note that $m_i \neq m_j$ for any $i, i' \in A_p$ with $i \neq i'$. Now it is easy to see that there exists an isomorphism of rings $R[M \times N] \rightarrow R[M][N]$ define by

$$
(a, (m_1, n_1), \ldots, (m_t, n_t)) \rightarrow (a, m_i(n_i)) R[M][N] \text{ for any } a \in R[M \times N].
$$
\[
\sum_{i=1}^{r} a_i(m_i, n_i) \rightarrow \sum_{p=1}^{r} \sum_{i \in A_p} (a, m_i) n_p.
\]

Suppose that
\[
(\sum_{i=1}^{r} a_i(m_i, n_i))(\sum_{j=1}^{s} b_j(m_j', n_j'))(\sum_{k=1}^{t} c_k(m_k'', n_k'')) = 0
\]
in \( R[M \times N] \). Then from the above isomorphism it follows that
\[
(\sum_{p=1}^{r} \sum_{i \in A_p} a_i m_i p)(\sum_{q=1}^{s} \sum_{j \in B_q} b_j m_j q)(\sum_{r=1}^{t} \sum_{k \in C_r} c_k m_k r) = 0.
\]

By Proposition 2.22, \( R[M] \) is a weak 3-\( N \)-Armendariz. Thus we have
\[
(\sum_{i \in A_p} a_i m_i)(\sum_{j \in B_q} b_j m_j)(\sum_{k \in C_r} c_k m_k) \in \text{nil}(R[M])
\]
for any \( p, q \) and \( r \). Since \( R \) is a weak 3-\( M \)-Armendariz, \( a b c \in \text{nil}(R) \) for any \( i \in A_p, j \in B_q \) and any \( k \in C_r \). Thus, \( a b c \in \text{nil}(R) \) for all \( i, j, k \). This means that \( R \) is a weak 3-(\( M \times N \))-Armendariz.

Let \( M_i, i \in I \), be monoids. Denote \( \bigotimes_{i \in I} M_i = \{(g_i)_{i \in I} \mid \text{there exist only finite } i \text{'s such that } g_i \neq e_i, \text{ the identity of } M_i \} \). Then \( \bigotimes_{i \in I} M_i \) is a monoid with the operation \((g_i)_{i \in I}(g'_i)_{i \in I} = (g_i g'_i)_{i \in I}\).

**Corollary 2.26** Let \( M_i, i \in I \) be u.p.-monoids and \( R \) a semicommutative ring. If \( R \) is a weak 3-\( M_0 \)-Armendariz for some \( i_0 \in I \), then \( R \) is a weak 3-\( \bigotimes_{i \in I} M_i \)-Armendariz.

**Proof.** Let \( \alpha = \sum_{i} a_i g_i, \beta = \sum_{j} b_j h_j, \gamma = \sum_{k} c_k l_k \in R[\bigotimes_{i \in I} M_i] \) such that \( \alpha \beta \gamma \in \text{nil}(R[\bigotimes_{i \in I} M_i]) \). Then \( \alpha, \beta, \gamma \in R[M_1 \times M_2 \times \cdots \times M_n] \), for some finite subset \( \{M_1, M_2, \cdots, M_n\} \subseteq \{M_i \mid i \in I\} \). Thus \( \alpha, \beta, \gamma \in R[M_{i_0} \times M_1 \times M_2 \times \cdots \times M_n] \). The ring \( R \), by Theorem 2.25 and by induction, is a weak 3-\( (M_{i_0} \times M_1 \times M_2 \times \cdots \times M_n) \)-Armendariz, so \( a b c \in \text{nil}(R) \) for all \( i, j, k \) and \( k \). Hence \( R \) is a weak 3-\( \bigotimes_{i \in I} M_i \)-Armendariz.

**Corollary 2.27** Let \( M \) be a monoid and \( R \) a semicommutative ring. If \( R \) is a weak 3-\( M \)-Armendariz, then \( R[x] \) and \( R[x, x^{-1}] \) are weak 3-\( M \)-Armendariz.

**Proof.** Note that \( R[x] \cong R[N \cup \{0\}] \) and \( R[x, x^{-1}] \cong R[Z] \).

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