FAILURE OF THE LIKELIHOOD RATIO METHOD: THE DISCRETE ARITHMETIC ASIAN CALL OPTION CASE

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ABSTRACT
Based on approximate analytical lower and upper bounds for the discrete arithmetic Asian call option vega, it is shown that this sensitivity cannot be obtained through application of the Monte Carlo likelihood ratio method. A discussion of alternative simulation methods is also included.

Keywords: Monte Carlo simulation, Asian call option, Greeks, Likelihood ratio method, Path-wise differentiation method, Finite difference method, Stochastic convex ordering.


1. INTRODUCTION
In finance mathematics the cost of a (self-financing) replication portfolio is given by an expectation with respect to a certain martingale measure (so-called risk neutral pricing method). It is not necessary to determine the replication portfolio itself, but once a pricing formula or pricing algorithm (e.g. Monte Carlo (MC) pricing algorithm) has been derived, then the replication portfolio may be expressed in terms of the partial derivatives of the price with respect to current model parameters, which are called sensitivities or Greeks. These quantities are important to assess the risk of a financial product (e.g. Fries [1], Chapter 7).

For complex products, like Bermudan options, an analytical formula is not available and the pricing must be done numerically. For higher dimensional models, like the LIBOR market model, the numerical method of choice is usually a MC simulation (e.g. Hürlimann [2]). For these reasons, the numerical calculation of sensitivities is in general done within the framework of Monte Carlo simulations, whatever the financial product is.

The main methods used for the calculation of Greeks are the finite difference method (FDM), the likelihood ratio method (LRM) and the path-wise differentiation method (PDM). Each of these has its own limitations. Despite its simple implementation, the FDM cannot be recommended. Though it delivers acceptable and generic market sensitivity estimates for continuous payoffs by small step sizes, it is problematic in dealing with discontinuous payoffs (extremely large Monte Carlo errors by small step sizes, e.g. [1], Section 15.4). Similarly, the PDM should not be used for an option whose payoff is discontinuous. Moreover, a major drawback of the PDM is the requirement of special knowledge about the payoff function and model realizations (e.g. [1], Section 15.5). On the other hand, the LRM does not require a smooth payoff function, but it has problems with smooth payoffs. In this situation the LRM can yield larger Monte Carlo errors than the FDM (a simple example is found in [1], Section 15.6.2).

In the present note, we apply the three main MC methods to determine the price, delta and vega sensitivities of a discrete arithmetic Asian call option. A detailed and rigorous comparison of these MC methods is undertaken. Section 2 summarizes the required formulas for Monte Carlo simulation. In general, results from MC simulation can be confirmed or rebutted through exact or approximate analytical methods provided they are available. Fortunately, the Asian call option has been the object of many investigations, and the results presented in Vanmaele et al. [3] exactly fit the present needs. Section 3 provides a brief summary of the used formulas. Then, based on a specific example, we show in Section 4 that the LRM fails. While the delta seems to converge with a slight overestimation and a somewhat erratic behaviour, the LRM does not produce a useful vega, which is overestimated by about 50%. In general, a reasonable way out of this inconvenience is the design of an approach that combines the well behaviour of the PDM for continuous payoffs and the LRM for discontinuous payoffs. Proposals in this direction are known and, to stimulate further research in this area, they are briefly reviewed.

2. PRICE AND GREEKS FOR THE DISCRETE ARITHMETIC ASIAN CALL OPTION
A European-style discrete arithmetic Asian call option is a financial instrument with exercise date \( T \), \( N \) equally spaced averaging dates at times \( k \cdot h, k = 1, \ldots, N, h = T / N \), and fixed strike price \( K \), which generates at maturity date \( T \) a financial payoff.
where \( x_i = \max(x,0) \) and \( S_t \) is the price of a risky asset at time \( t \in [0,T] \). Clearly, the risk neutral price of this call option at the current time \( t = 0 \) is given by

\[
P(N,K,T) = e^{-rT} \cdot E^Q \left[ \left( N^{-1} \cdot \sum_{k=1}^{N} S_{k,h} - K \right)_+ \right]
\]  

for a martingale measure \( Q \) and a given risk-neutral interest rate \( r \). We assume a Black-Scholes-Merton return model such that the price of the risky asset is described by

\[
S_t = S_0 \cdot \exp \left( (r - \frac{1}{2} \sigma^2) t + \sigma \sqrt{t} \cdot Z_t \right),
\]

where \( (Z_t) \) is a Wiener process. For ease of notation the average price of a risky asset is denoted by \( \bar{S} = N^{-1} \cdot \sum_{k=1}^{N} S_{k,h} \). The corresponding discounted path-dependent payoff is denoted by

\[
g(S_{k,h};k=1,...,N) = g(\bar{S}) = e^{-rT} \cdot \left( \bar{S} - K \right)_+.
\]

Recall briefly the needed formulas for MC simulation of the Greeks, which are based on the finite difference method (FDM), the path-wise differentiation method (PDM) and the likelihood ratio method (LRM) (consult e.g. Glasserman [4], Section 7). We use the discretization step \( h = T/N \) in the Euler scheme for the logarithmic random variable \( X_k = \ln(S_{k,h}) \), \( k = 1,...,N \), such that

\[
X_0 = \ln(S_0), \quad X_k = X_{k-1} + (r - \frac{1}{2} \sigma^2)h + \sigma \sqrt{h} \cdot Z_k, \quad k = 1,...,N,
\]

where \( Z_1, Z_2, ..., Z_N \) are independent standard normal variables. In the numerical illustration of Section 4, we base the MC simulation on 20 batches of \( R = 1'000 \) simulation paths each and use their antithetic counterparts. For the parameter choice \( T = 1, N = 250 \), of equally spaced daily transactions over one year, the MC simulation results in a total number of 5 million independent draws of standard normal random numbers (= 20·250·1'000 = 5000000) and their antithetic counterparts. As a notational convenience, we add the superscript “a” to MC simulated values that are obtained from the set of antithetic random variables \( Z_1^a = -Z_1, Z_2^a = -Z_2, ..., Z_N^a = -Z_N \).

### 2.1. Simulation paths and MC price of the Asian call option

For simplification, denote the drift in (2.3) by \( \mu = r - \frac{1}{2} \sigma^2 \). The simulation paths \( i = 1,...,R \), and their antithetic counterparts, as well as the MC price, are specified as follows. The simulation paths are given by

\[
X_{i,0} = \ln(S_0), \quad X_{i,k} = X_{i,k-1} + \mu \cdot h + \sigma \sqrt{h} \cdot Z_{i,k}, \quad k = 1,...,N, \quad i = 1,...,R.
\]

The Monte Carlo price of the discrete arithmetic Asian call option is determined by

\[
E[g(\bar{S})] = \frac{1}{2R} \cdot \sum_{i=1}^{R} \left( g(\bar{S}_i) + g(\bar{S}_i^a) \right)
\]

\[
\bar{S}_i = N^{-1} \cdot \sum_{k=1}^{N} e^{X_{i,k}}, \quad \bar{S}_i^a = N^{-1} \cdot \sum_{k=1}^{N} e^{X_{i,k}^a}, \quad i = 1,...,N.
\]
2.2. Finite difference method (FDM)

For the calculation of the MC delta and vega sensitivities, one needs up and down shifts of the simulated paths by quantities $\pm e, e = 1\%$, and their antithetic counterparts, for which superscript notations “u” and “d” are used. The FDM centered values of delta and vega are defined as follows. For the delta sensitivity, one has

$$\begin{align*}
X_{i,0}^u &= \ln(S_0) + \ln(1 + e), \\
X_{i,k}^u &= X_{i,k-1}^u + \mu \cdot h + \sigma \sqrt{h} \cdot Z_{i,k}, \\
X_{i,0}^{a,u} &= \ln(S_0) + \ln(1 + e), \\
X_{i,k}^{a,u} &= X_{i,k-1}^{a,u} + \mu \cdot h - \sigma \sqrt{h} \cdot Z_{i,k}, \\
X_{i,0}^{d} &= \ln(S_0) + \ln(1 - e), \\
X_{i,k}^{d} &= X_{i,k-1}^{d} + \mu \cdot h + \sigma \sqrt{h} \cdot Z_{i,k}, \\
X_{i,0}^{d,a} &= \ln(S_0) + \ln(1 - e), \\
X_{i,k}^{d,a} &= X_{i,k-1}^{d,a} + \mu \cdot h - \sigma \sqrt{h} \cdot Z_{i,k}, \quad k = 1, \ldots, N, \quad i = 1, \ldots, R.
\end{align*}$$

(2.8)

$$\frac{\partial \hat{E}[g(S_I)]}{\partial S_0} \approx \frac{1}{4 \cdot e \cdot S_0 \cdot R} \sum_{i=1}^{N} \left[ g(S_{i,u}^u) - g(S_{i,d}^d) + g(S_{i,u}^{a,u}) - g(S_{i,d}^{a,d}) \right].$$

(2.9)

For the vega sensitivity, with the shifted drifts and volatilities $\mu_u = r - \frac{1}{2} \sigma_u^2, \sigma_u = \sigma + e, \mu_d = r - \frac{1}{2} \sigma_d^2, \sigma_d = \sigma - e$, use is made of the formulas

$$\begin{align*}
X_{i,0}^u &= \ln(S_0), \\
X_{i,k}^u &= X_{i,k-1}^u + \mu_u \cdot h + \sigma_u \sqrt{h} \cdot Z_{i,k}, \\
X_{i,0}^{a,u} &= \ln(S_0), \\
X_{i,k}^{a,u} &= X_{i,k-1}^{a,u} + \mu_u \cdot h - \sigma_u \sqrt{h} \cdot Z_{i,k}, \\
X_{i,0}^{d} &= \ln(S_0), \\
X_{i,k}^{d} &= X_{i,k-1}^{d} + \mu_d \cdot h + \sigma_d \sqrt{h} \cdot Z_{i,k}, \\
X_{i,0}^{d,a} &= \ln(S_0), \\
X_{i,k}^{d,a} &= X_{i,k-1}^{d,a} + \mu_d \cdot h - \sigma_d \sqrt{h} \cdot Z_{i,k}, \quad k = 1, \ldots, N, \quad i = 1, \ldots, R.
\end{align*}$$

(2.10)

$$\frac{\partial \hat{E}[g(S_I)]}{\partial \sigma} \approx \frac{1}{2 \cdot e \cdot R} \sum_{i=1}^{N} \left[ g(S_{i,u}^u) - g(S_{i,d}^d) + g(S_{i,u}^{a,u}) - g(S_{i,d}^{a,d}) \right].$$

(2.11)

2.3. Path-wise differentiation method (PDM)

In this situation, it is possible to exchange differentiation and expected values. The MC evaluation depends on the simulated paths defined in (2.6) and (2.7). With [4], Example 7.2.2, one obtains for the delta sensitivity the MC estimator

$$\frac{\partial \hat{E}[g(S)]}{\partial S_0} = \hat{E}\left[ \frac{\partial}{\partial S_0} g(S) \right] = \hat{E}\left[ e^{-\tau} \cdot 1[S > K] \left( \frac{S}{S_0} \right) \right].$$

(2.12)

Similarly, with [4], Example 7.2.3, one obtains for the vega sensitivity...
\[
\frac{\partial}{\partial \sigma} E[g(S)] = E\left[ \frac{\partial}{\partial \sigma} g(S) \right] = E\left[ e^{-rT} G(S) \cdot \frac{\partial S}{\partial \sigma} \right] = \frac{e^{-rT}}{2 \cdot R \cdot \sum_{i=1}^{g} \left( 1 \right)} \frac{\partial S}{\partial \sigma} + \frac{\partial S^\prime}{\partial \sigma},
\]

\[
\frac{\partial S}{\partial \sigma} = N^{-1} \sum_{k=1}^{N} \left\{ X_{i,k} - \ln(S_0) - (r + 0.5\sigma^2) \cdot h \cdot k \right\},
\]

\[
\frac{\partial S^\prime}{\partial \sigma} = N^{-1} \sum_{k=1}^{N} \left\{ X_{i,k}^\prime - \ln(S_0) - (r + 0.5\sigma^2) \cdot h \cdot k \right\}.
\]

### 2.4. Likelihood ratio method (LRM)

Given arbitrary financial payoffs, it is possible to differentiate probabilities rather than the payoffs. For the joint probability \( p(S_{kh}; k = 1, ..., N) \) of the price process in (2.3) one obtains from [4], Example 7.3.2, the delta sensitivity

\[
\frac{\partial}{\partial S_0} E[g(S)] = E\left[ g(S) \cdot \frac{\partial \ln p(S_{kh}; k = 1, ..., N)}{\partial S_0} \right] = E\left[ g(S) \cdot \frac{Z_1}{S_0 \cdot \sigma \cdot \sqrt{h}} \right]
\]

\[
\approx \frac{1}{2 \cdot S_0 \cdot \sigma \cdot \sqrt{h} \cdot R} \sum_{i=1}^{g} \left( g(S_i) - g(S_i^\prime) \right) \cdot Z_{i,1}
\]

Similarly, with [4], Example 7.3.3, one calculates the vega sensitivity as follows:

\[
\frac{\partial}{\partial \sigma} E[g(S)] = E\left[ g(S) \cdot \frac{\partial \ln p(S_{kh}; k = 1, ..., N)}{\partial \sigma} \right] = E\left[ g(S) \cdot N \cdot \frac{Z^2 - 1}{\sigma} - \sqrt{h} \cdot Z \right]
\]

\[
\approx \frac{N}{2 \cdot R} \sum_{i=1}^{g} \left( g(S_i) \cdot \left( \frac{Z^2 - 1}{\sigma} - \sqrt{h} \cdot Z_i \right) + g(S_i^\prime) \left( \frac{Z^2 - 1}{\sigma} + \sqrt{h} \cdot Z_i \right) \right)
\]

### 3. ANALYTICAL APPROXIMATION FORMULAS

Results from MC simulation can be confirmed or rebutted through exact or approximate analytical methods provided they are available. Fortunately, the Asian call option has been the object of many investigations. We rely on the so-called comonotonic approximation method, which consists to bound price and Greeks of financial instruments using appropriate lower and upper bounds. It is based on stochastic ordering convex approximations derived from comonotonic random sums extensively discussed since Kaas et al. [5] and Dhaene et al. [6], and successfully applied in many other papers (e.g. Vanduffel et al. [7], [8], Hürlimann [9], [10]). All formulas below refer to the results presented in Vanmaele et al. [3]. Instead of days we measure time in years.

#### 3.1. Price upper bound

From [3], Theorem 6, formula for UBGAd one gets

\[
P_{UB} = P_{LB} + N^{-1} \cdot e^{-rT} \cdot \sigma(d),
\]

\[
d = \frac{6N}{(N + 1)(2N + 1)} \left\{ \frac{1}{4} \sigma h(N + 1) + \frac{\ln(K/S_0) - rh(N + 1)}{\sigma} \right\}
\]

\[
h(d) = \sum_{i=0}^{N} \sum_{j=0}^{N} q_{i,j} \Phi(d - \sigma \cdot (\rho_{N-i,j} \sqrt{T - ih} + \rho_{N-j} \sqrt{T - jh}))
\]

\[
q_{i,j} = \exp \left\{ \sigma \left[ \min(T - ih, T - jh) - \rho_{N-i,j} \sqrt{T - ih} \sqrt{T - jh} \right] \cdot (c_{i,j} - 1) \right\}
\]

\[
c_{i,j} = \exp \left\{ \sigma \left[ \min(T - ih, T - jh) - \rho_{N-i,j} \sqrt{T - ih} \sqrt{T - jh} \right] \right\}
\]
3.2. Delta (approximate) lower and upper bound

From [3], Table 6, one borrows the lower bound

\[ \Delta_{LB} = N^{-1} \sum_{j=0}^{N-1} e^{-rT} \cdot d_{i,j} \]  

and the upper bound

\[ \Delta_{UB} = \Delta_{LB} + \frac{e^{-rT}}{2N} \cdot \sqrt{\Phi(d) \cdot h(d)} \cdot \{1 + \eta(d)\}, \]

\[ \eta(d) = -\frac{1}{2\sigma} \cdot \frac{6N}{(N+1)(2N+1)} \cdot \varsigma(d), \quad \varsigma(d) = \frac{\varphi(d)}{\Phi(d)} + \frac{1}{h(d)} \cdot \frac{\partial h}{\partial d}, \]

\[ \frac{\partial h}{\partial d}(d) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} q_{ij} \cdot (d - \sigma \cdot (\rho N_i - \sqrt{T - jh}) \]  

\[ (3.4) \]

3.3. Vega (approximate) lower and upper bound

Again, Table 6 in [3] yields the lower bound

\[ V_{LB} = \frac{e^{-rT}}{N} \cdot \sum_{j=0}^{N-1} K_j \cdot \rho_{N-j} \cdot \sqrt{T - jh} \cdot \varphi(d) \cdot \Phi^{-1}(x_{i,K}) \]  

and the upper bound

\[ V_{UB} = V_{LB} + \frac{e^{-rT}}{2N} \cdot S_0 \cdot \sqrt{\Phi(d) \cdot h(d) \cdot \varsigma(d) \cdot \frac{\partial d}{\partial \sigma}}, \]

\[ \frac{\partial d}{\partial \sigma} = \sqrt{\frac{6N}{(N+1)(2N+1)}} \cdot \left\{ \frac{1}{4} \cdot h(N+1) + \frac{1}{2} \cdot r h(N+1 - \ln(K/S_0)) \right\} / \sigma^2 \]  

\[ (3.5) \]

\[ (3.6) \]

\[ (3.7) \]

4. NUMERICAL ILLUSTRATION

Our numerical illustration is based on the parameter choice

\[ T = 1, N = 250, S_0 = K = 100, r = 5\%, \sigma = 20\% . \]

The MC simulation is made of 20 batches of \( R = 1000 \) simulation paths each, and their antithetic counterparts are also used. The MC results are compared in Table 1 with the exact and approximate bounds.

| Price and Greeks for the discrete arithmetic Asian call option. | | |
|---|---|---|---|
| MC simulation | exact analytical bounds |
| Price | 5.80467 | 5.80387 | 5.78160 | 5.82614 |
| Delta | 0.59574 | 0.58862 | 0.58914 | 0.58931 | 0.58917 | 0.58945 |

The MC values with the FDM and PDM converge satisfactorily, but this is not the case for the LRM. While the delta seems to converge with a slight overestimation and a somewhat erratic pattern, the LRM does not produce a useful vega value, which is overestimated by about 50%. The Figures 1 and 2 illustrate these facts. In general, it is known that the PDM yields smaller MC standard errors than the LRM. For a more thorough discussion of these properties consult Glasserman [4], Section 7.
5. CONCLUSIONS AND FUTURE WORK

To conclude and to stimulate further research in this area, let us briefly review some work, which has been proposed to overcome the presented failure. A first choice is the adjoint-likelihood method, which combines the PDM with the LRM. Building on the ideas of L’Ecuyer [11], [12] on hybrid path-wise/LRM sensitivity calculations, this approach has been developed by Giles [13] under the naming “Vibrato Monte Carlo Sensitivities”. The method applies the path-wise approach to the differentiable path simulation and uses the LRM approach for the discontinuous path evolution. The use of Malliavin calculus can also be viewed as a hybrid path-wise/LRM combination, as shown by Chen and Glasserman [14]. As stated in Section 4 of Giles [13], the vibrato approach is completely compatible with an adjoint (or backward) calculation of the path sensitivity (in contrast to a forward standard path-wise calculation). It is therefore possible to obtain an unlimited number of first order sensitivities to input parameters (such as initial price, interest rate, volatility, etc.) at a cost which is similar to the cost of the original calculation. The adjoint implementation of the vibrato approach is nowadays called adjoint/likelihood method.

Concerning implementation of the adjoint-likelihood method, we recommend the original article [13] and the M.Sc. thesis of Keegan [15]. The multilevel Monte Carlo extension by Burgos and Giles [16] is also of interest (see also the PhD thesis by Burgos [17]). It must be emphasized that the development of adjoint codes is based on (generic) ideas of Algorithmic Differentiation (AD) (e.g. Capriotti [18], [19], Capriotti and Giles [20], Homescu [21], Safiran
[22], and Chan and Joshi [23]). Therefore, automated AD tools should also be used. The adjoint/AD approach is also reviewed in the “opus magna” on interest rate derivatives by Andersen and Piterbarg [24].

6. REFERENCES