ANALYSIS OF A SINGULAR PARABOLIC PARTIAL DIFFERENTIAL EQUATION BY USING ALGEBRAIC APPROACHES

Alfred Huber
Innovation Centre W.E.I.Z., Franz-Pichler-Straße 30, A-8160 Weiz
Email: alfred.huber@innovationszentrum-weiz.at

ABSTRACT
In this paper we suggest different algebraic approaches to analyze the solution structure (or solution manifold) of a nonlinear singular parabolic partial differential equation (nPDE) importantly in applications of theoretical economics. The crucial step is the fact that the solution manifold of a general nPDE under consideration can be expressed in terms of special functions which are solutions of a specific (nonlinear) first-order ordinary differential equation (nODE), e.g. the Riccati equation. By using algebraic methods derived by the author we further show how one can augment the solution manifold in order to get different classes of solutions. Then we analyze and interpret (if possible) the meaning of these solutions.

Key words: Nonlinear partial differential equations (nPDEs), nonlinear ordinary differential equations (nODEs), algebraic solution techniques, special function approach, homogeneous balance method.
PACS-Code: 02.30Jr, 02.20Qs, 02.30Hq,
AMS-Code: 35L05, 35Q53, 14H05.

1. Introduction

In this paper we deal with the following nPDE (and its variations)

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial x}\left(1+\frac{1}{u^2}\right) - u^2\right\} \tag{1}$$

looking for the unknown function $u = u(x,t)$ in $(x,t) \in \Omega_T \coloneqq R^+ \times (0,T)$, where $T > 0$ and $R^+ = \{0,T\}$. In this context the unknown function $u = u(x,t)$ is related to the Arrow-Pratt coefficient of absolute risk aversion for the optimal value function [1]. We assume that the function $u = u(x,t)$ is positive and non-increasing. The eq.(1) undertakes the following conditions: $u \geq 0$, $\frac{\partial u}{\partial x}(0,t) = 0$ and $u(x,t) \to \alpha$ as $x \to \infty$, where α is a non-negative constant. The above given problem is related by the initial condition $u(x,0) = u_0(x)$ upon $x \in R^+$, where the non-increasing initial value u_0 belongs to $H^1(R^+)$. The authors in [2] solved the eq.(1) in a weak sense. In the following we show how one can get different classes of solutions by using algebraic methods.

Note: We arrange to suppress the item 'classes of solutions' although this is the correct term; so we just mean solutions instead of classes of solutions. A further remark may be useful: There are several views in literature relating to the order of the time derivative; nPDEs of the form $u_{t,t} = K[u_x, u_{xx}, ...]$ with similar higher-order mixed time derivations are also called evolution equations (EVEs).

2. Some general notes

Although the origin of nPDEs (in the general form of an EVE $u_t = K[u, u_x, u_t, u_{xx}, u_{xt}, ...]$ where $K[u]$ means a nonlinear operator in general and the function $u = u(x,t)$ is sought) is very old, they have undergone remarkable new developments during the last half of the 20th century. One of the main impulses for developing nPDEs have been the study of nonlinear wave propagation problems (e.g. the Dutch Korteweg and de Vries in 1834 and the French Boussinesq). We stress, since a general theory does not exist, alternative methods are indispensable (on the contrary in the linear case one can make use of Fourier-, Green- and Laplacian transformations) for deriving solutions.

Also, as an essential statement we stress that nature is nonlinear a priori (e.g. gravitational fields, heat and/or diffusion propagation, magnetic fields, nonlinear chemical reactor and many others – linearization considerably simplifies the problem). These problems arise in different areas of applied mathematics, physics, and engineering sciences including fluid dynamics [3], nonlinear optics [4], solid state mechanics [5], plasma physics [6], astrophysics [7], biological sciences relating to the Davydov soliton [8] and medical sciences [9], [10] or physical chemistry [11] to mention some examples of practical interest.

Pure mathematics like geometry [12], differential geometry including curved surfaces [13] are also related to nPDEs (maybe of higher order than two).

However, as mentioned above, in contrast with linear wave theory, one is confronted here with a large variety of 'sophisticated methods'. The most used methods are the tanh-approach [14], the mixing exponential method [15] and

the Weierstraß transform method [16] and [17] to mention some of them. More analytical solutions can be found in [18], [19] and [20].
A purely algebraic approach, the HBM (the **H**omogeneous **B**alance **M**ethod) [21], and their improvements can be found in [22] and [23].
A method using series of pure sine and cosine functions [24] as well as the method of the Jacobian elliptic functions (and also the Weierstraß elliptic function – the $\wp$-function), e.g. [25], [26], [27], [28] and [29] are also useful to generate solutions.
It is worth to mention further standard methods for obtaining solutions, e.g. the Hirota's bilinear method [30], the inverse scattering transform IST [31], a modified Sine(h)-Gordon-Method, [32], the Painlevé expansion [33], the Lie group analysis (LGA) used by the author [34] and [35] - classical as well as non-classical LGA were used successfully. Another approach derived by the author can be found in [36]. All these methods are suitable to get a deeper insight into the solution manifold of nPDEs, e.g. [37], [38], [39], [40], [41] and [42].
They are the only tool for creating solutions in an easy way and they deliver exact results in reasonable time and computational effort. In some cases however the order, respectively the kind of the nonlinearity causes troubles and moreover, cases in which algebraic methods fail, are known (in such cases the only successful way is to use standard numerical methods or simply some series expansions in the form of Taylor and/or Laurent).
For further reading 'classical ansatz-methods' are discussed intensively in [43], [44] and [45].
Similarity reductions dealing with Lie symmetries and invariant properties are also appropriate to calculate solutions as shown by the author's recent papers, e.g. [34], [35], [36] and [37].
Difficulties will be explained later but due to the experiences of the author one can say that such algebraic methods work well in the most cases. They represent the only rational way to generate solutions of nPDEs as well as the relating nODEs (especially in cases of higher order).

3. Basics

For the following we agree: The first calculation is given in detail; after that we only give the 'ansatz' as well as the result.

Note: There is no exact translation for the German word 'ansatz' (sing.) and 'ansätze' (plur.) - a starting point is meant.
Consider a nPDE in its two independent variables x and t, which describes the dynamical evolution of a wave form, $u(x,t)$, $u:\Omega \to R$ in the domain $\Omega \subseteq R^d$, $d \geq 2$ and time t, so that the mapping:

$$P\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial x^2},\frac{\partial u}{\partial x}\frac{\partial u}{\partial t},\ \ldots\ldots,\frac{\partial^{n-1} u}{\partial x^{n-1}},\frac{\partial^n u}{\partial x^n}\right)=0 \tag{2}$$

for $u:\Omega\times R^+ \to R$, $R^+ = \{t \in R : t > 0\}$, $\Omega \subseteq R^d$ holds. The extension to an arbitrary number of independent variables should not be difficult. Firstly the nPDE, eq.(2) is converted (the travelling-wave reduction or equivalently a similarity transformation) into a nODE by using a frame of reference, $u(x,t) \to f(\xi)$ and λ is a constant to be determined later (in general wave theory or soliton propagation λ means the velocity). Thus one has

$$Q\left(f(\xi), f'(\xi), \ldots\ldots, f^{(n-1)}(\xi), f^{n}(\xi)\right) = 0\,, \tag{3}$$

where the prime denotes differentiation w.r.t. ξ. $f(\xi)$ represents a localized wave solution and exemplifies a stationary wave with characteristic width $L = \lambda^{-1}$. The nODE, eq.(3) is integrated as long as all terms contain derivatives. Further the associated constants of integration can be set to zero in view of the localized solutions we seek. This is a necessary (but not sufficient) condition that $f(\xi)$ tends to zero as $\xi \longrightarrow \pm\infty$.

Note: This is important if we are interested in solitary solutions as the meaning as a solitary boundary condition. In general one may not omit or set it equal to zero as it happens in some papers. This is only allowed in context of soliton theory. For further generalization it may be useful to introduce a constant α such that $\xi = \alpha\, x - \lambda\, t$ holds.
The next step is that the solution can be expressed in terms of the following series representation by using an auxiliary variable $w = w(\xi)$ such that the function $u(x,t) = f(\xi)$ can be represented in terms of a series like

$$u(x,t) = f(\xi) = \sum_{i=0}^{n} a_i w^i(\xi) \tag{4}$$

in which the coefficients a_i will be calculated as solutions of a (in general) nonlinear algebraic system of polynomial equations. The parameter 'n' in eq.(4) is found by balancing the highest derivative with the nonlinear terms in the reduced nODE, eq.(3). This parameter must be a positive integer since it represents the number of terms in the series eq.(4).

Note: In the case of fractions one can take transformations as shown in [46]. In this study the nonlinear heat conduction equation was analyzed leading to the case $n = -1$.
Next, one assumes that the function $w(\xi)$ satisfies a Riccati equation with the constant k as

$$\frac{dw}{d\xi} = w' = k(1 \pm w^2). \qquad (5)$$

Substituting (2) and (3) in the relevant nODE will yield a system of nonlinear polynomial equations with respect to $a_0, a_1, \ldots, k, \alpha$ and λ. Special solutions of (5) are expressed as

$$w(\xi) = \tanh(k\xi) \quad \text{and/or} \quad w(\xi) = \coth(k\xi)\,. \qquad (6)$$

$w(\xi)$ can also be taken as $w(\xi) = \frac{e^{2k\xi}+e^{2c_1}}{e^{2k\xi}-e^{2c_1}}$ or with some simplifications as $w(\xi) = \coth(k\xi - c_1)$, where c_1 means an arbitrary constant. Another possible representation of $w(\xi)$ is the form $w(\xi) = 1/(1 + e^{k\xi})$, but using this method, only tanh-type solutions can be obtained.
For enlarging the set of solutions, Riccati's equation can be used in the form

$$w'(\xi) = 1 + w(\xi)^2\,, \qquad (7)$$

with the prime denoting differentiation w.r.t. ξ once again. The solutions depend upon the sign of k leading to solutions given in [38]. Instead of ξ one can also set $(\xi - \xi_0)$ with arbitrary chosen ξ_0 . Thus by using different nODEs one is able to enlarge the solution manifold.

4. Intermediate results

First, the following knowledge is important: By reduction of the nPDE eq.(1), the nonlinear algebraic polynomial equations does not yield a nontrivial solution,
so the nPDE eq.(1) cannot be solved and the hyperbolic tangent method fails (we got $a_0 = a_1 = k = \lambda = 0$). The next attempt was to use a generalization of the 'ansatz' (4) in the form $u(x,t) \to f(\xi) = a_0 + a_1 v(\xi) + a_2 v(\xi)^2 + \frac{b_0}{v(\xi)} + b_1 + b_2 v(\xi)$ and using the nODE eq.(5). This also did not show success. Of course it could have happened that the parameter 'n' in the series eq.(4) disappears, but that did not happen in this case. Other nODEs did also not provide any meaningful solutions e.g. $v' = av(\xi) + bv(\xi)^2 + cv(\xi)^3$ (an Abelian-like equation).
Thus, following [2] by performing the transformation $u(x,t) \to u + \alpha,\ \alpha \in \mathbb{R}^+ \setminus \{0\}$, a further nPDE results:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial x}\left(1 + \frac{1}{(u+\alpha)^2}\right) - (u+\alpha)^2\right\} \qquad (8)$$

considering the (initial) conditions for the function $u(x,t)$: $\frac{\partial u}{\partial x}(0,t) = 0$,
$\frac{\partial u}{\partial x}(x,t) \le 0$ for $(x,t) \in \Omega_T$ and $u \to 0$ as $x \to \infty$ for $0 < t < T$ and $u(x,0) = u_0(x) - a$ for $x \in \mathbb{R}^+$. Multiplying out eq.(8) and applying the travelling wave reduction in the form $u(x,t) \to f(\xi) = x - \lambda\, t$ the following nODE results (the nODE consists of 16 nonlinear terms; for control purposes we write down the first and last terms):

$$f^3 f'' + 3\,\alpha f^2 f'' + 3\,\alpha^2 f f'' + \cdots + 3\alpha^2\,\lambda f f' + \alpha^3\,\lambda f' = 0, \qquad (9)$$

for the function $f = f(\xi)$. The balancing procedure yields $n = 1$, so the linear ansatz $f(\xi) = a_0 + a_1 w$ is of advantage. As a nODE we used a Riccati equation like $w' = k(1 - w^2)$.
This yields a nonlinear algebraic system of polynomial equations consisting of seven equations with four unknowns and thus we got two solutions as

*) $\alpha = \frac{1}{2}(\lambda + 2a_0)$, a_1, k arbitrary, and
**) $\alpha = -a_0$, a_1, k, λ arbitrary. (10)

A further calculation, using a hyperbolic angular series in the form $f(\xi) = a_0 + \sum_{i=1}^{n}(a_i coshw + b_0 sinhw) sinhw^{i-1}$ yields, for $n = 1$, the linear ansatz $f(\xi) = a_0 + a_1 coshw + b_1 sinhw$. For the nODE we used $w' = sinhw$. Inserting this we also found a nontrivial solution of the nonlinear system of polynomial equations (11 equations for 5 unknowns, say $a_0, a_1, b_1, \alpha, \lambda$):

*) $\alpha = i - a_0$ with condition $\alpha + a_0 \neq 0$ and a_1, b_1, λ arbitrary, i is the imaginary unit
**) $\lambda = (\alpha + a_0)$, a_1, b_1 arbitrary. (11)

Note: One try, using the linear ansatz and as a nODE of the first order $w'(\xi) = (Aw - a)(Bw - b)$ only leads the trivial solution. Moreover, applying the Jacobian method, say $f(\xi) = \sum_{i=1}^{n} sn^{i-1}(\xi)$ and as the nODE $w'(\xi) = d.sn(w,k)$ fails – here also the trivial solution results. As a further remarkable note we assume the following:
In [2] the solution set of the eq.[8] is assumed for $x \in \mathbb{R}^+$. For enlarging the solution manifold we assume $x \in \mathbb{R}^\pm$ and moreover, since $\alpha = i - a_0$, with the imaginary unit $x \in \mathbb{C}$!

Next we deal with the following nPDE [2] looking for the unknown function $u(x,t) \epsilon E^L$

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}\left(1+\frac{1}{(s+\alpha)^2}\right)-\frac{2}{(s+\alpha)^3}\left(\frac{\partial u}{\partial x}\right)^2-2(u+\alpha)\frac{\partial u}{\partial x}\ , \qquad (12)$$

in $(x,t)\epsilon\Omega_T^L$ with $s\epsilon E^L$ and E^L (a convex set) defined in [2]. This equation can be derived from the eq.(1) after some transformations.

Note: A standard theory of PDEs tells us that an a-priori bound of u and $\partial u/\partial x$ ensures the existence of solutions e.g. [47]. However, it must be checked exactly in every case whether this is also applicable for nonlinear equations. For the given nPDE, the eq.(1), this will be examined in detail in a next study.
After some transformations, the 'plane wave ansatz' $u(x,t)\to f(\xi)=x-\lambda t$ is done. Introducing a new dependent variable by $p(\xi)$, since $n=-1$ and $f(\xi)=p(\xi)^{-1}$ a new nODE for the function $p(\xi)$ results:

$$2(\alpha+s)^3pp'+2\alpha(\alpha+s)^3p^2p'-\lambda(\alpha+s)^3p^2p'-2p'^2+2(\alpha+s)^3p'^2p+$$
$$+2(a+s)p'^2p-(\alpha+s)^3p^2p''-(\alpha+s)p^2p''=0. \qquad (13)$$

Here, for the nODE eq.(13) the unusual case $n=0$ appears by balancing, so that the application of the previous solution methods is not possible. It is best to try a (Taylor) series solution around $\xi=0$ resulting in

$$p(\xi)=1+\xi+a_2\xi^2+\frac{1}{6}\xi^3(4a_2^2+2a_2-1)+\frac{1}{24}\xi^4(18a_2+32a_2^2+8a_2^3-$$
$$-8a_2(2a_2+3)-1)+\cdots\sigma[\xi]^5, \qquad (14)$$

with special values for $a_0=a_1=1$ and a_2 arbitrary but unequal zero.
Next, a further nPDE resulting from some transformations as shown in [2] is

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+\frac{\left(\frac{\partial u}{\partial x}\right)^2}{\frac{\partial^2 u}{\partial x^2}}=0, \qquad (15)$$

looking for the unknown function $u=u(x,t)$ whereby the second derivative may not vanish.
Using the 'plane wave ansatz' by $u(x,t)\to f(\xi)=x-\lambda t$, yielding the nODE
$\lambda f'f''+f''^2-f'^2=0$, once again looking for the unknown function $f(\xi)$.
Again, the already occurring several times effect occurs that the balancing parameter disappears, that is $n=0$, so that the balancing procedure fails (more exactly balancing the above given nODE yields $n=-1$, thus we used the linear ansatz $f(\xi)=p(\xi)^{-1}$ for the new function $p(\xi)$. The nODE for this function (only the first and the last terms) is $p^2p''^2-\cdots.-p^2p'^2=0$. Balancing this nODE we see that the parameter remains undetermined.
Note: It is noteworthy that the nPDE, eq.(1) and its variants very often show this result of balancing, namely that the parameter 'n' disappears or remains indeterminate – we cannot explain this 'unusual' behavior at this time.
But an easy solution can be obtained by the following procedure: A transformation $f'(\xi)=w(\xi)$ yields a nODE for the function $w(\xi)$ in the form

$$\lambda ww'+w'^2-w^2=0\ , \qquad (16)$$

which is integrated exactly at once by

$$w(\xi)=exp\left[\frac{1}{2}\left(-\lambda\pm\sqrt{4+\lambda^2}\right)\xi\right]. \qquad (16.a)$$

To find the function $f(\xi)$ one has to integrate the expression (16.a) resulting in

$$f(\xi)=\int w(\xi)d\xi=\pm\frac{2exp\left[-\frac{1}{2}\left(\lambda+\sqrt{4+\lambda^2}\right)\xi\right]}{-\lambda+\sqrt{4+\lambda^2}}. \qquad (16.b)$$

As a last nPDE we analyze the following [2]:

$$\frac{\partial^2 u}{\partial x\partial t}-\left(1+\frac{1}{(s+a)^2}\right)\frac{\partial^3 u}{\partial x^3}-\frac{2}{(s+a)^3}\left(\frac{\partial^2 u}{\partial x^2}+2\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2}-2(u+a)\frac{\partial^2 u}{\partial x^2}-$$
$$-2\left(-\frac{3}{(s+a)^3}\frac{\partial^2 u}{\partial x^2}+1\right)\left(\frac{\partial u}{\partial x}\right)^2=0, \text{ looking for the function } u=u(x,t). \qquad (17)$$

Using the 'plane wave ansatz' $u(x,t)\to f(\xi)=x-\lambda t$ once again, one has to handle the following highly nODE of the third order for the function $f(\xi)$:

$$\left(1+\frac{1}{(s+a)^2}\right)f'''+\lambda f''+\frac{2}{(s+a)^3}(f''+2f')f''+2(f+a)f''+$$
$$+2\left(-\frac{3}{(s+a)^3}f''+1\right)f'^2=0, \qquad (18)$$

where s and α are some constants.
The next step is to apply the balancing procedure leading to the value of the balancing parameter $n=1$, thus the linear ansatz of the form $f(\xi)=a_0+a_1v(\xi)$ is suitable. For the underlying nODE we used a Riccati-Equation in the form $v'(\xi)=k(1-v(\xi)^2)$.
Performing the steps given above we have seven polynomial equations in total with four unknowns. But here another surprise appears: The solution of the algebraic system of polynomial equations admits only the trivial solution, that is

in detail $a_1 = k = 0$. And further that means that the application of the hyperbolic tangent method fails and we have to look for alternative 'ansätze' to handle the eq.(18).
Next, trying to solve the eq.(18), we used 'ansätze' containing Lambert functions as well as an improved exponential series, but both of the methods fail since the set of polynomial equations admits only trivial solutions.
A successful way was given by the following: For the function $f(\xi)$ we used the 'ansatz' $f(\xi) = \sum_{i=0}^{1} a_i v^i(\xi) + \sum_{i=0}^{1} b_i v^{i-1}(\xi)$, and this gives for the case $n = 1$ the linear form $f(\xi) = a_0 + a_1 v(\xi) + \frac{b_0}{v(\xi)} + b_1$. For the nODE we used an Abelian-like equation in the form $v'(\xi) = A_0 + A_1 v(\xi) + A_2 v^2(\xi) + A_3 v^3(\xi)$. In total we have 17 nonlinear polynomial equations for 9 unknowns, that are $a_0, a_1, \ldots\ldots, A_0, \ldots\ldots, \lambda$. The solution manifold consists of 13 solutions whereby only one of them is suitable, that is in detail:

*) $A_0 = A_1 = A_3 = 0$, A_2 arbitrary, $a_1 = 0$, a_0, b_1, b_0, λ arbitrary and $A_2 b_0 \neq 0$. (19)

5. Function analysis – discussion of the function's behavior

We start with the solutions for the coefficients in eq.(10). These solutions are given – after introducing the coefficients from above by

$$f_1(\xi) = \frac{1}{2}(2a - \lambda) + a_1 \left(\frac{e^{2k\xi} - e^{2c_1}}{e^{2k\xi} + e^{2c_1}}\right), \text{ with } a, \lambda, c_1, k, a_1 \text{ arbitrary and } e^{2k\xi} + e^{2c_1} \neq 0, \quad (20)$$

$$f_2(\xi) = -a + a_1 \left(\frac{e^{2k\xi} - e^{2c_1}}{e^{2k\xi} + e^{2c_1}}\right), \text{ with } a, \lambda, c_1, k, a_1 \text{ arbitrary and } e^{2k\xi} + e^{2c_1} \neq 0. \quad (20.a)$$

A graphical overview of both functions is given by the Fig.1 and Fig.2., also a three-dimensional plot of these functions is seen.
Using the coefficients given in eq.(11) we have an imaginary solution:

$$f_3(\xi) = (i - a) + a_1 cosh\left[2arctanh\left[e^{c_1 + x}\right]\right] + b_1 sinh\left[2arctanh\left[e^{c_1 + x}\right]\right]. \quad (20.b)$$

For the second solution we have a real-valued expression (with a real-valued pre-factor):

$$f_4(\xi) = (\lambda - a) + a_1 cosh\left[2arctanh\left[e^{c_1 + x}\right]\right] + b_1 sinh\left[2arctanh\left[e^{c_1 + x}\right]\right]. \quad (20.c)$$

Both functions are plotted and can be seen in the Fig.3 and Fig.4.
To examine the function $f_5(\xi)$ given by the coefficients of eq.(19) we have the explicit representation:

$$f_5(\xi) = a_0 - b_0(A_2\xi + c_1) + b_1, \quad (20.d)$$

with arbitrary chosen values for the coefficients a_0, b_0, b_1, A_2 and c_1 whereby this function is seen in the Fig.5.

5.a Some series representations

For practical calculations we write down some series with lower order (for all functions we used special values, e.g. $c_1 = a_1 = a = k = \lambda = 1$):

$$f_1(\xi) \approx \frac{3 - e^2}{2(1 + e^2)} + \frac{4e^2\xi}{(1 + e^2)^2} + \frac{4(e^4 - e^2)\xi^2}{(1 + e^2)^3} + O[\xi]^3, \quad (21)$$

$$f_2(\xi) \approx -\frac{2e^2}{(1 + e^2)} + \frac{4\,e^2\xi}{(1 + e^2)^2} + \frac{4(e^4 - e^2)\xi^2}{(1 + e^2)^3} + O[\xi]^3, \quad (21.a)$$

and for the complex-valued function (20.b) we derive a series of the form $\sim(a - bi)$ with special values of the parameters $c_1 = 0,\ a_1 = a = k = \lambda = 1$:

$$f_3(\xi) \approx (i - 1) + \cos[(\pi + iln2 - iln\xi + O[\xi]^2) - i\sin[(\pi + iln2 - iln\xi + O[\xi]^2). \quad (21.b)$$

The radii of convergence should be determined separately. A same expression can be derived for the function $f_4(\xi)$. We explicitly have

$$f_4(\xi) = (\lambda - a) + a_1 cosh[2\ arctanh[e^{x + c_1}] + b_1 sinh[2\ arctanh[e^{x + c_1}]. \quad (21.c)$$

This expression can be converted into pure exponential terms by

$$f_4(\xi) = 1 + \frac{1}{1 + e^x} + \frac{e^x}{1 - e^x}, \quad (21.d)$$

where we have used special values by $\lambda = 2, a = 1, a_1 = b_1 = 1$ and $c_1 = 0$. For this real-valued function we have

$$f_4(\xi) \approx 1 - \frac{2}{\xi} - \frac{\xi}{6} + O[\xi]^2. \quad (21.e)$$

Finally, for the last function we have

$$f_5(\xi) \approx (a_0 + b_1 - c_1 b_0) - A_2 b_0 \xi + O[\xi]^3. \quad (21.f)$$

5.b Some Limit behavior

Here, for practical computations, we give values either for $\xi \to 0$ and $\xi \to \infty$, they are explicitly

$$\lim_{\xi \to 0} f_1(\xi) = \frac{3 - e^2}{2 + 2e^2} \approx -0{,}26, \qquad \lim_{\xi \to \infty} f_1(\xi) = \frac{3}{2},$$

$$\lim_{\xi \to 0} f_2(\xi) = -\frac{2e^2}{1 + e^2} \approx -1{,}76, \qquad \lim_{\xi \to \infty} f_2(\xi) = \infty,$$

$$\lim_{\xi\to 0} f_3(\xi) = (-3+i)\,, \qquad \lim_{\xi\to\infty} f_3(\xi) = (-2+i)\,,$$
$$\lim_{\xi\to 0} f_4(\xi) = -\infty\,, \qquad \lim_{\xi\to\infty} f_4(\xi) = 0\,,$$
$$\lim_{\xi\to 0} f_5(\xi) = 1\,, \qquad \lim_{\xi\to\infty} f_5(\xi) = -\infty\,, \tag{22}$$

where 'i' means the imaginary unit.

5.c Asymptotic Stability

Theorem: The kink solutions (21) and (21.a) are stable asymptotically.

Proof: The asymptotical stability of the solutions eq.(21) and eq.(21.a) can be shown by perturbation analysis. We show this only for the function $f_1(\xi)$:
Introduce $r = x - ct$ into (21) and assume a perturbed solution as $u(r,t) = u(r,c) + \varepsilon\, v(r,t)$
where the second term represents a small perturbation with ε as a perturbation parameter.
One says that a solution $u(r,t)$ is stable if either $\lim_{t\to\infty} v(r,t) = 0$ or $\lim_{t\to-\infty} v(r,t) = v_r(r,c)$.

Put $v(r,t) = f(r)e^{-\lambda t}$ and let λ be an eigenvalue to derive an eigenvalue problem for $f(r)$.
From the standard results on eigenvalue problems one can conclude that all eigenvalues are real and positive and hence all perturbations of the solutions have a finite extent decay exponentially
in time for $t \to \infty$ in this case. Therefore the kink solutions (21) and (21.a) are asymptotically stable □.
Note: Take a look to the function $f_4(\xi)$, eq.(21.c): In view of heat transport theory, linear combinations of hyperbolic functions allow one to interpret this classes of solutions as a damped motion without heat sources independently of the choice of the parameter values. Further take a look to the solution eq.(20). We explicitly have, since $\lambda = 1$:

$$f_1(\xi) = u_1(x,t) = \left(\frac{e^{2(x-t)} - e^2}{e^{2(x-t)} + e^2}\right) + \frac{1}{2}\,. \tag{23}$$

The function is bounded and does not vanish as $t \to \infty$ which can be seen from (22). Such classes of functions are suitable to describe the decay of temperature curves by considering suitable boundary conditions. A similar behavior can be expected from the solution eq.(21.a)

6. Summary, open problems, future intensions

In this paper, new classes of solutions of a (2+1) nonlinear evolution equation of the second order by using direct algebraic approaches could derive. The nPDE under consideration takes place in theoretical economic studies.
It is well known that classes of solutions of nonlinear evolution equations can be expressed as finite series in terms of special functions, e.g. tanh-functions, Weierstrass and Jacobian functions.
All these different algebraic approaches are useful if one assumes the possibility to balance the nonlinear term and the highest linear term leading to a suitable number (a positive integer) in the series expression eq.(4). This number may not be equal to zero since it represents the number of terms in the series in this expansion (nevertheless some equations allow $n = 0$).
The Sine-Gordon Equation is a notable exception, although their solutions contain hyperbolic functions; a similar example is the nonlinear Schrödinger-Equation.
Another limitation is the solubility of the relating homogeneous nonlinear algebraic system.
Let us now mentioned a few words to different 'ansätze' used in the 'exact reduction' process generating nODEs from nPDEs. Commonly used 'ansätze' are $u(x,t) = U(x - \lambda t)$, acting as a reference frame and $u(x,t) = U\left(\frac{x}{3t}\right)^{1/3}$ especially for 'diffusion-like' parabolic equations. These 'ansätze', resulting from Lie group analysis (LGA) are appropriate to reduce a nPDE to a nODE in the sense of a similarity reduction. Not only class of travelling wave solutions occur, moreover, classes of general solutions can be obtained. However, difficulties appear in solving the relating nODE derived from the similarity reduction (for some nODEs only a numerical procedure is appropriate).
Note: Using the diffusion-like 'ansatz' one gets the highly nODE for the function $u = u(\xi)$, with $\xi = x.t^{-1/3}$

$$3u'' + 3u''^{u^{-\frac{2}{3}}} - 2u'^2 u - 4u'^2 u^{-\frac{5}{3}} - 18tu^{\frac{1}{3}}u' + 9xu' = 0, \tag{24}$$

Where the prime means differentiation w.r.t. to ξ. A numerical calculation is plotted in Fig.8 by using the initial values $u(0) = 1$ and $u'(0) = -1$.

On the contrary, this is the main advantage of algebraic procedures since they allow to generate classes of solutions in the same manner but without solving complicate nODEs.

Comparing the power of algebraic approaches with LGA one can see that they work as an excellent alternative tool without using any numerical methods. Apart from trivial solutions of the homogeneous nonlinear algebraic system in some cases it may happen that such systems could be solved only numerically.
Otherwise, one can make use of the freedom of the appearing constants. In fact, it is clear that by increasing the number of constants the solubility process of the nonlinear algebraic system can therefore be influenced by the user.
As a last remark one should point out the possibility in translating the algebraic procedure in any computer languages.
As a next study we will analyse the solution structure of the nPDE, eq.(1) by using Lie Group Analysis and comparing the new results by the given ones.

7. Figures

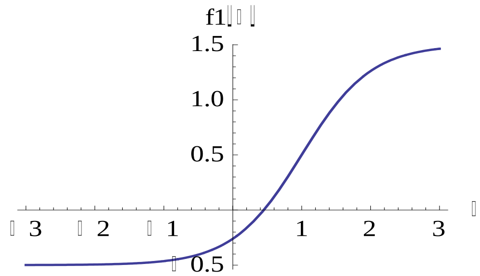

Fig.1 A typical kink-like behavior of the solution function $f_1(\xi)$ with special values of the constants $a = a_1 = c_1 = \lambda = k = 1$. The function is asymptotically stable as mentioned in 5.c.

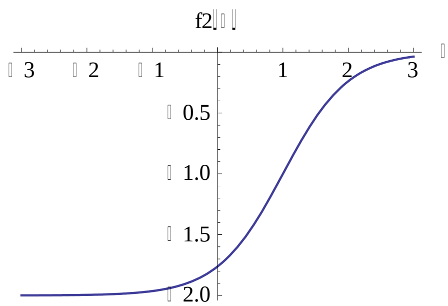

Fig.2 A further typical kink-like behavior of the solution function $f_2(\xi)$ with special values of the constants $a = a_1 = c_1 = \lambda = k = 1$, moving upon the vertical axes. This kink is also asymptotical stable.

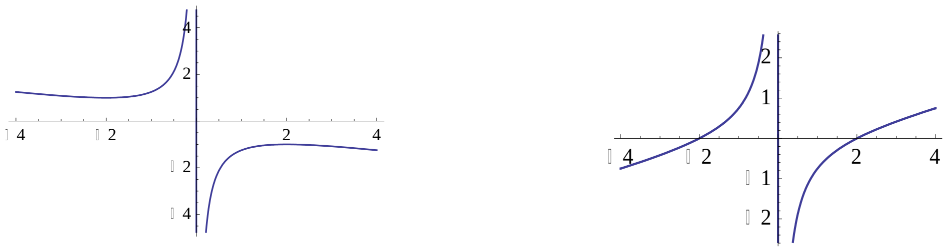

Fig.3
The real part (left) and the imaginary part (right) of the solution function $f_3(\xi)$ with special values of the parameter $a = a_1 = b_1 = 1$. Clearly one can see the discontinuity.

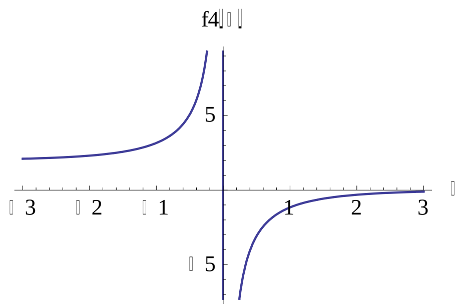

Fig.4 The solution function $f_4(\xi)$ generated by different values of the constants, that are $\lambda = 2, a = 1, c_1 = 0,\ a_1 = b_1 = 1$. Also in this sketch one can see clearly the discontinuity

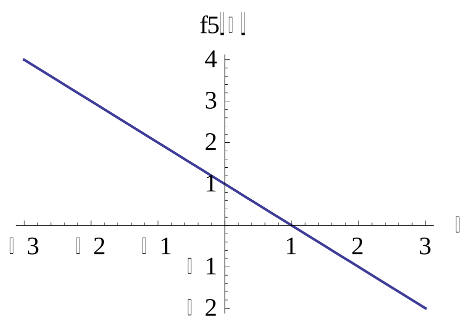

Fig.5 The linear behavior of the solution function $f_5(\xi)$ with special values of the constants $a = a_1 = c_1 = \lambda = k = 1$. The function is also stable.

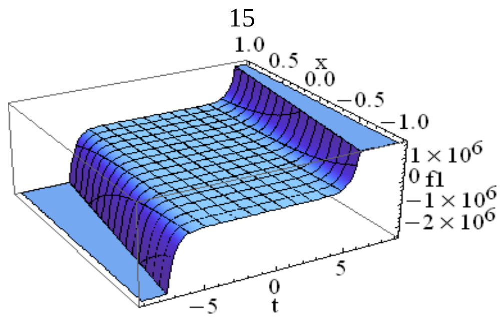

Fig.6 A three-dimensional plot of the function $f_1(\xi)$

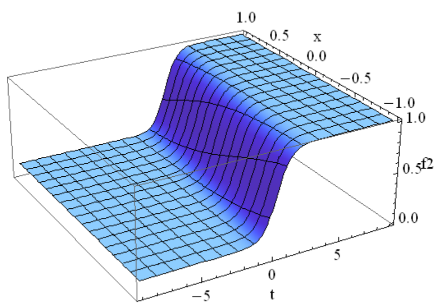

Fig.7 A three-dimensional plot of the function (the kink-solution) $f_2(\xi)$

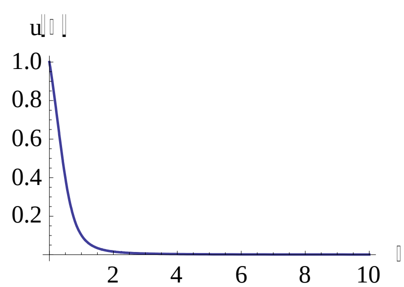

Fig.8 A plot of the nPDE, eq.(24) for the function $u(\xi)$ with the initial conditions $u(0) = 1$ and $u'(0) = -1$.

8. References

[1]. J. W. Pratt, Risk aversion in the small and in the large, Econometrica 32, p.112, 1964
[2]. R. Abe, N. Ishimura, Existence of solutions for the nonlinear partial differential equation arising in the optimal investment problem, Proc. Japan Acad. 84, Ser. A, p.11, 2008
[3]. G. Witham, Linear and Nonlinear Waves, Wiley, N.Y., 577, 1974
[4]. H. Steudel, Solitons in Stimulated Raman Scattering, Exp. Tech. Phys. 20, p.409, 1972
[5]. G. Eilenberger, Solitons, Springer Verlag, Berlin, 140, 1983
[6]. A. Hasegawa, Plasma Instabilities and Nonlinear Effects, Springer, Berlin, 190, 1975
[7]. C. C. Lin, Origin of the magnetic spiral arms in the galaxy NGC 6946, Astrophys. J. 140, p.646, 1964
[8]. S. Takeno, Vibron Solitons and Coherent Polarization in an Exactly Tractable Oscillator-Lattice System, Prog. Theoret. Phys. 73, p.853, 1985
[9]. S. Yomosa, Solitary excitations in deoxyribonuclei acid (DNA) double helices, Phys. Rev. A 30, p.474, 1984
[10]. S. Yomosa, Solitary Waves in Large Blood Vessels, J. Phys. Soc. Jpn. 56, p.506, 1987
[11]. P. Gray, S. Scott, Chemical Oscillations and Instabilities, Clarendon, Oxford, p.292, 1992
[12]. R. C. Jurgensen, Geometry, 5th ed., McDougal, 2000
[13]. M. P. Do Carmo, Differential Geometry of Curves and Surfaces, 2nd ed., Dover, 2016
[14]. [14] W. Malfliet, Solitary wave solutions of nonlinear wave equations, Amer. J. Phys. 60, p.650, 1992
[15]. J. Parkes, An automated tanh-function method for finding solitary wave solutions to nonlinear evolution equations, Comp. Phys. Commun. 98, p.288, 1996
[16]. A. Huber, A novel class of solutions for a non-linear third order wave equation generated by the Weierstraß transform, Chaos, Solitons and Fractals 28/4, p.972, 2000
[17]. A. Huber, The calculation of novel class of solutions of a nonlinear fourth order evolution equation by the Weierstraß transform method, Appl. Math. and Comp. 201, p.668, 2008
[18]. S. A. Elwakil, S. K. El-labany, Exact travelling wave solutions for the generalized shallow water equation, Chaos, Solitons, Fractals 17/1, p.121, 2003
[19]. S. A. Elwakil, S. K. El-labany, New exact solutions for a generalized variable coefficient 2D KdV equation, Chaos, Solitons, Fractals 19/5, p.1083, 2004
[20]. S. A. Elwakil, S. K. El-labany, Higher-order solution of an electron acoustic solitary waves via vortex electron distribution, Chaos, Solitons, Fractals 22/1, p.13, 2004
[21]. M. Wang, Exact solutions for a compound KdV-Burgers equation, Phys. Lett. A 213, p.279, 1996
[22]. E. Fan, H. Zhang, New exact solutions to a system of coupled KdV equations, Phys. Lett. A 245, p.389, 1998
[23]. E. Fan, H. Zhang, A note on the homogeneous balance method, Phys. Lett. A 246, p.403, 1998
[24]. D. Cao, New exact solutions for a class of nonlinear coupled differential equations, Phys. Lett. A 296, p.27, 2002
[25]. H. Chen, H. Zhang, Improved Jacobian elliptic function method and its applications, Chaos, Solitons and Fractals 15, p.585, 2003
[26]. H. Zhang, New exact Jacobi elliptic function solutions for some nonlinear evolution equations, Chaos, Solitons and Fractals 32, p.653, 2005
[27]. Z. Fu, S. Liu, Q. Zhao, New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations, Phys. Lett. A 290, p.72, 2001
[28]. J. Parkes, The Jacobi-elliptic function method for finding periodic-wave solutions to nonlinear evolution equations, Phys. Let. A 295, p.280, 2002
[29]. J. Nickel, Travelling wave solutions to the Kuramoto-Sivashinsky equation, Chaos, Solitons and Fractals 33, p.1376, 2007
[30]. P. Drazin, R. Johnson, Solitons: An Introduction, Cambridge Univ. Press, N.Y., 1989
[31]. analogues [30]
[32]. Z. Yan, A New Sine-Gordon Equat. Expansion Algorithm to Investigate some special Nonlinear Differential Equations, MM Res. Preprint, Acad. Sinica 23, p.300, 2004
[33]. analogues to [30] and [31]
[34]. A. Huber, A note on class of travelling wave solutions of a nonlinear third order system generated by Lie's approach, Chaos, Solitons and Fractals 32, p.1357, 2007
[35]. A. Huber, A note on class of solitary and solitary-like solutions of the Tzitzéica-equation, Physica D 237, p.1079, 2008
[36]. A. Huber, A novel algebraic procedure for solving nonlinear evolution equations of higher order, Chaos, Solitons and Fractals 34/3, p.765, 2007

[37]. A. Huber, On class of solutions of a Nonlinear Differential Equation of Sixth Order generated by a Classical and Non-Classical Symmetry Analysis, Int. J. of Differential Equations and Dynamical Systems, Vol. 15, Nos.1 & 2, p.27, 2007
[38]. A. Huber, Solitary solutions of some nonlinear evolution equations, Appl. Math. and Comp. 166/2, p.464, 2005
[39]. analogues to [35]
[40]. A. Huber, A novel algebraic algorithm for solving evolution equations of higher order by use of Lambert functions, Int. J. of Res. and Rev. in Appl. Sci. 3 (1), p.47, 2010
[41]. A. Huber, A novel algebraic procedure for solving evolution equations of higher order, Chaos, Solitons and Fractals 34/3, p.765, 2007
[42]. analogues to [17]
[43]. A. A. Darwish, Applications of algebraic method to exactly solve some nonlinear partial differential equations, Chaos, Solitons and Fractals 33/4, p.1263, 2007
[44]. S. A. Khuri, Traveling wave solutions for nonlinear differential equations: A unified ansätze approach, Chaos, Solitons and Fractals 32/1, p.252, 2007
[45]. J. Nickel, Travelling wave solutions to the Kuramoto–Sivashinsky equation, Chaos, Solitons and Fractals 33, p.1376, 2007
[46]. A. Huber, On nonlinear evolution equations of higher order – the introduction and application of a novel comput. approach, Appl. Math. and Comp**.** 215, p.2337, 2009
[47]. A. Friedman, Partial Differential Equations of Parabolic type, Krieger Publ., 1983
[48]. P. L. Sachdev, Nonlinear Diffuse Waves, first ed., Cambridge University Press, (1987)