SOME TECHNIQUES FOR SOLVING FREDHOLM-VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND

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ABSTRACT
In this paper, the existence and uniqueness of solution of the linear Fredholm- Volterra integral equation **(LF-VIE)** of the second kind with continuous kernel is discussed and proved, then we use a numerical method to reduce this type of equations to a system of Volterra integral equations.RungeKutta method **(R.KM)**and Block by block method **(Bby BM)** are used to solve a system of linear Volterra integral equations **(SLVIEs)**of the second kind with continuous kernel. Numerical examples are considered to illustrate the effectiveness of the proposed methods and the error is estimated.

Keywords: *Fredholm- Volterra Integral Equation; Runge-Kutta method; Block by block method.*

1. Introduction

Many problems in engineering, science and mathematical physics lead to integral equation [1]. There are many well-written researchon the theory and applications of integral equations in different sciences.Among these research, the research which were studied numerical solution of many types of integral equations, such as the Toeplitz matrix method, the product Nyström method, the Galerkin method; (**R.KM)** and (**BbyBM)** (see Linz [2], Baker et al. [3], and Delves and Mohamed [4]). More informations for some numerical methods can be found especially in Delves and Mohamed [4], Atkinson [5, 6] and Golberg [7]. In [8], **(BbyBM)** was used to solve the system of nonlinear Volterra integral. In [9], the authors solved the Fredholm-Volterra integral equation of the second kind usingdiscreteAdomian decomposition method. In [10],the authors obtained numerically the solution of class of two-dimensional Fredholm-Volterra integral equation by Collocation method. In [11], Hendi and Albugamisolved Fredholm-Volterra integral equation by using Collocation and Galerkin methods. In this paper, we use **(R.KM)** and (**BbyBM)** to discuss numerically the solution of a **(SVIEs)**which is obtained by using a numerical method on the **(F-VIE)**of the second kind with continuous kernel of the form

$$u(x,t)-\lambda\int_a^b k(x,y)u(y,t)dy-\lambda\int_0^t F(t,\tau)u(x,\tau)d\tau=f(x,t) \qquad (1)$$

kernels $k(x,y)$, $F(t,\tau)$ are continuous functions, where $k(x,y)$ is the kernel of Fredholm part, and $F(t,\tau)$ is the kernel of the Volterra part respectively. The function $f(x,t)$ is the free term of the integral equation, $u(x,\tau)$ is the unknown function and λ is a constant has a physical meaning.

2. Existence and unique the solution:

In order to guarantee the existence of a unique solution of equation (1), we assume the following conditions:

i) The kernel of the Fredholm term satisfies the continuity condition:

$\left|k(x,y)\right|\le N_1, \quad \forall 0\le x\le b$, ($N_1$ is a constant)

ii) The kernel of the Volterra term $F(t,\tau)\in C([0,\mathrm{T}]\times[0,\mathrm{T}]), \forall t\in[0,\mathrm{T}]$,satisfies:

$\left|F(t,\tau)\right|\le N_2$,($N_2$ is a constant)

iii) The given function $f(x,t)$ with its derivatives with respect to x and t are continuous in $\mathrm{L}_2[a,b]\times\mathrm{C}[0,\mathrm{T}]$ where,

$$\left\|f(x,t)\right\|=\max_{0\le t\le T}\int_0^t\left[\int_a^b\left|f(x,\tau)\right|^2dx\right]^{\frac{1}{2}}d\tau=N_3 \qquad ,(N_3 \text{ is a constant})$$

iv) The unknown function $u(x,t)$ will be discussed in the space $\mathrm{L}_2[a,b]\times\mathrm{C}[0,\mathrm{T}],\mathrm{T}<\infty$, and in this space it behaves as the known function $f(x,t)$, where

$$\|u(x,t)\| = \max_{0\le t\le T}\int_0^t [\int_a^b u^2(x,\tau)dx\,]^{\frac{1}{2}} d\tau = B \qquad ,(B \text{ is a constant})$$

Now, we prove that the solution is exist using the successive approximation method, also, called the Picard method, that we pick up any real continuous function $u_0(x,t)$ in $L_2[a,b]\times C[0,T]$,

Then construct a sequence u_n defined by

$$u_n(x,t) = f(x,t) + \lambda\int_a^b k(x,y)u_{n-1}(y,t)dy + \lambda\int_0^t F(t,\tau)u_{n-1}(x,\tau)d\tau$$

$$u_{n-1}(x,t) = f(x,t) + \lambda\int_a^b k(x,y)u_{n-2}(y,t)dy + \lambda\int_0^t F(t,\tau)u_{n-2}(x,\tau)d\tau$$

For ease of manipulation it is convenient to introduce:

$$v_n(x,t) = u_n(x,t) - u_{n-1}(x,t)$$
$$= \lambda\int_a^b k(x,y)[u_{n-1}(y,t) - u_{n-2}(y,t)]dy + \lambda\int_0^t F(t,\tau)[u_{n-1}(x,\tau) - u_{n-2}(x,\tau)]d\tau$$

where , $n = 1, 2, \ldots$

Then

$$u_n(x,t) = \sum_{i=0}^{n} v_i(x,t) \qquad (2)$$

Hence, $v_n(x,t) = \lambda\int_a^b k(x,y)v_{n-1}(y,t)dy + \lambda\int_0^t F(t,\tau)v_{n-1}(x,\tau)d\tau$

Using the properties of the norm we obtain:

$$\|v_n(x,t)\| \le |\lambda| \left\|\int_a^b k(x,y)v_{n-1}(y,t)dy\right\| + |\lambda| \left\|\int_0^t F(t,\tau)v_{n-1}(x,\tau)d\tau\right\|$$

For $n = 1$, we get

$$\|v_1(x,t)\| \le |\lambda| \left\|\int_a^b k(x,y)v_0(y,t)dy\right\| + |\lambda| \left\|\int_0^t F(t,\tau)v_0(x,\tau)d\tau\right\|$$
$$\le |\lambda| \left\|\int_a^b |k(x,y)||v_0(y,t)|dy\right\| + |\lambda| \left\|\int_0^t |F(t,\tau)||v_0(x,\tau)|d\tau\right\|$$

Then from conditions (i),(ii) and (iii) with $v_0 = f(x,t)$ and $\|f\| = N_3$, we get

$$\|v_1(x,t)\| \le |\lambda| N_1 \int_a^b \|v_0(y,t)\| dy + |\lambda| N_2 \int_0^t \|v_0(x,\tau)\| d\tau$$
$$\le |\lambda| N_1 N_3 + |\lambda| N_2 N_3 \|t\|$$

But, we have $0 \le \tau \le t \le T$, then $\max|t| = T$, then, we have

$$\|v_1(x,t)\| \le |\lambda| N_3 (N_1 + N_2 T)$$

In general, we get

$$\|v_1(x,t)\| \le |\lambda|^n N_3 (N_1 + N_2 T)^n = N_3 \alpha^n \ , \qquad \alpha = |\lambda| (N_1 + N_2 T) \tag{3}$$

This bound makes the sequence $v_n(x,t)$ converges if

$$\alpha < 1 \Rightarrow |\lambda| < \frac{1}{N_1 + N_2 T} \tag{4}$$

The result (4), leads us to say that the formula (2) has a convergent solution, where

$$u_n(x,t) = \sum_{i=0}^{n} v_i(x,t) = N_3 \sum_{i=0}^{n} \alpha^i \tag{5}$$

So, let $n \to \infty$,we have

$$u(x,t) = \sum_{i=0}^{\infty} v_i(x,t) = \frac{N_3}{1-\alpha} \ , \quad (\alpha < 1) \tag{6}$$

The infinite series of (6) is convergent, and $u(x,t)$ represents the convergent solution of equation (1). Also each of v_i is continuous, therefore $u(x,t)$ is also continuous.

To show that $u(x,t)$ is unique, we assume that $\tilde{u}(x,t)$ is also a continuous solution of (1) then, we write

$$\tilde{u}(x,t) = \lambda \int_a^b k(x,y) \tilde{u}(y,t) dy + \lambda \int_0^t F(t,\tau) \tilde{u}(x,\tau) d\tau + f(x,t) \tag{7}$$

Then, we get

$$u(x,t) - \tilde{u}(x,t) = \lambda \int_a^b k(x,y) [u(y,t) - \tilde{u}(y,t)] dy + \lambda \int_0^t F(t,\tau) [u(x,\tau) - \tilde{u}(x,\tau)] d\tau$$

Then, applying Cauchy-Schwarz inequality, we get

$$\|u(x,t) - \tilde{u}(x,t)\| \le |\lambda| \left\| \int_a^b |k(x,y)| |u(y,t) - \tilde{u}(y,t)| dy \right\| + |\lambda| \left\| \int_0^t |F(t,\tau)| |u(x,\tau) - \tilde{u}(x,\tau)| d\tau \right\|$$

Using conditions (i) and (ii), we obtain:

$$\|u(x,t)-\tilde{u}(x,t)\| \le |\lambda| N_1 \int_a^b \|u(y,t)-\tilde{u}(y,t)\| dy + |\lambda| N_2 \int_0^t \|u(x,\tau)-\tilde{u}(x,\tau)\| d\tau$$

Where $0 \le \tau \le t \le T$, $\max|t| = T$, we get

$$\|u(x,t)-\tilde{u}(x,t)\| \le |\lambda| N_1 \|u(y,t)-\tilde{u}(y,t)\| + |\lambda| N_2 T \ \|u(x,\tau)-\tilde{u}(x,\tau)\|$$
$$\le \alpha \|u(x,t)-\tilde{u}(x,t)\|$$

We obtain that

$$(1-\alpha)\|u(x,t)-\tilde{u}(x,t)\| \le 0$$

When $\|u(x,t)-\tilde{u}(x,t)\|$ is positive and $\alpha < 1$, we get:

$$\|u(x,t)-\tilde{u}(x,t)\| = 0 \Rightarrow u(x,t) = \tilde{u}(x,t) \tag{8}$$

Then it has a unique solution.

3. The (SLVIEs)

Consider the linear integral equation:

$$u(x,t) = f(x,t) + \lambda \int_a^b k(x,y) u(y,t) dy + \lambda \int_0^t F(t,\tau) u(x,\tau) d\tau \tag{9}$$

When $a = b = 0$, equation (9) becomes:

$$u(0,t) = f(0,t) + \lambda \int_0^t F(t,\tau) u(0,\tau) d\tau$$

Then,

$$u_0(t) = f_0(t) + \lambda \int_0^t F(t,\tau) u_0(\tau) d\tau \tag{10}$$

Where $u_0(t) = u(0,t), \quad f_0(t) = f(0,t)$.

The formula (10) represents Volterra integral equation of the second kind at $a = b = 0$.

For representing (9) as a**(SVIEs)**, we use the numerical method. Divide the interval $[a, b]$ as $a = x_0 \le x_1 \le ... \le x_N = b$. Using the quadrature formula, the integral equation (9) becomes:

$$u_i(t) - \lambda \int_0^t F(t,\tau) u(x_i,\tau) d\tau = g_i(t)$$

where $g_i(t) = f_i(t) + \lambda \sum_{n=0}^{i} w_n k_{in} u_n(t)$

Then,

$$u_i - \lambda \int_0^t F(t,\tau) u(x_i,\tau) d\tau = g_i + E \qquad , i = 1,2,...,N \tag{11}$$

where $E = |u_i - u(x,t)|$.

The formula (11) leads us to say that, we have N unknown functions $u_i(t)$ corresponding to the interval $[\mathrm{a,b}]$, when $\mu =$ constant $\neq 0$, this is a **(SVIEs)**of the second kind, if $\mu = 0$ it is of the first kind.

Now, we will use **(R.KM)**and (**BbyBM)**for solving a **(SLVIEs).**

3.1. The (R.KM)

In this section, the **(R.KM)**is used to solve **(LF-VIE)**of the second kind. By divide the interval $[\mathrm{a,b}]$ as $a = x_0 \le x_1 \le ... \le x_i \le ... \le x_N = b$, $i = 0,1,...,N$ and using the quadrature formula, the integral equation (1) represent a **(SVIEs)**as:

$$u_i(t) - \lambda \int_0^t F(t,\tau) u(x_i,\tau) d\tau = g_i(t)$$

Where $g_i(t) = f_i(t) + \lambda \sum_{n=0}^{i} w_n k_{in} u_n(t)$

To solve the **(SLVIEs)**:

$$U(t) - \lambda \int_0^t F(t,\tau) U(\tau) d\tau = F(t) + \lambda \sum_{j=0}^{i} w_j k_{nj} U(t) \qquad , n = 0,1,...N \tag{12}$$

Where,

$$U(t) = (u_1(t), u_2(t), ..., u_l(t))^T, U(\tau) = (u_1(\tau), u_2(\tau), ..., u_l(\tau)),$$
$$F(t) = (f_1(t), f_2(t), ..., f_l(t))^T$$

And

$$F(t,\tau)U(\tau) = \begin{bmatrix} F_{1,1}(t,\tau)U(\tau) & F_{1,2}(t,\tau)U(\tau) & \dots & F_{1,s}(t,\tau)U(\tau) \\ F_{2,1}(t,\tau)U(\tau) & F_{2,2}(t,\tau)U(\tau) & \dots & F_{2,s}(t,\tau)U(\tau) \\ \vdots & \vdots & \ddots & \vdots \\ F_{s,1}(t,\tau)U(\tau) & F_{s,2}(t,\tau)U(\tau) & \dots & F_{s,s}(t,\tau)U(\tau) \end{bmatrix}$$

Then, we get

$$u_n(t) - \lambda \int_0^t F(t,\tau) U(\tau) d\tau = g_n(t) \qquad , n = 0,1,...,N, \tag{13}$$

Now, applying the **(R.KM)** for solve the integral equations (13)

Suppose that:

$$F(t,\tau)=\sum_{s}\phi_s(t)\chi_s(\tau) \tag{14}$$

Substituting from (14) into (13),

$$u_n(t)=g_n(x)+\int_0^t\sum_{s}\phi_s(t)\chi_s(\tau)U(\tau)d\tau$$

$$u_n(t)=g_n(x)+\sum_{s}\phi_s(t)\int_0^t\chi_s(\tau)U(\tau)\,\mathrm{d}\tau$$

Then, we have,

$$u_n(t)=g_n(t)+\sum_{s}\phi_s(t)z_s(t),t>0 \tag{15}$$

where,

$$z_s(t)=\int_0^t\chi_s(\tau)U(\tau)\,\mathrm{d}\tau \tag{16}$$

By derivative (16), we have,

$$z_s'(t)=\chi_s(\tau)U(\tau), \qquad\qquad z_s(0)=0$$

Now, apply the **(R.KM)** to this system of equations to give,

$$\tilde{z}(v_ph)=h\sum_{q=0}^{p-1}A_{pq}\chi_s(v_qh)\tilde{U}(v_qh)\ \ p=1,2,...,m$$

Which lead to,

$$\tilde{u}_n(v_ph)=g_n(v_ph)+\sum_{s}\tilde{z}_s(v_ph)\phi_s(v_ph)$$

$$=g_n(v_ph)+\sum_{s}\phi_s(v_ph)h\sum_{q=0}^{p-1}A_{pq}\chi_s(v_qh)\tilde{U}(v_qh)$$

$$=g_n(v_ph)+h\sum_{q=0}^{p-1}A_{pq}\tilde{U}(v_qh)\sum_{s}\phi_s(v_ph)\chi_s(v_qh)$$

By using equation (14) to give,

$$\tilde{u}_n(v_ph)=g_n(v_ph)+h\sum_{q=0}^{p-1}A_{pq}F(v_ph,v_qh)\tilde{U}(v_qh)\ ,\quad p=1,2,...,m \tag{17}$$

which is approximate solution for equation (13).

Now, if $m = 4$ consider the Pouzet's derivation, we define:

$$
\begin{aligned}
&p_{i,j}(t) = u_{i,j-1}(t) \\
&q_{i,j}(t) = G_{i,j}(t_{j+\frac{1}{2}}) + \frac{h}{2}F(t_{j+\frac{1}{2}}, t_j)p_{i,j} \\
&r_{i,j}(t) = G_{i,j}(t_{j+\frac{1}{2}}) + \frac{h}{2}F(t_{j+\frac{1}{2}}, t_{j+\frac{1}{2}})\mathrm{q}_{i,j} \\
&s_{i,j}(t) = G_{i,j}(t_{j+1}) + hF(t_{j+1}, t_{j+1})\mathrm{r}_{i,j} \\
&u_{i,j}(t) = G_{i,j}(t_{j+1}) + \frac{h}{6}[F(t_{j+1}, t_j)p_{i,j} + 2F(t_{j+1}, t_{j+\frac{1}{2}})[\mathrm{q}_{i,j} + \mathrm{r}_{i,j}] + F(t_{j+1}, t_{j+1})\mathrm{s}_{i,j}
\end{aligned}
\tag{18}
$$

The function $u_{i,j}(t)$ is unknown function, such that

$$G_{i,j}(t) = g_i(t) + \frac{h}{6}\sum_{n=0}^{j-1}[F(t,t_n)p_{i,n} + 2F(t,t_{n+\frac{1}{2}})[\mathrm{q}_{i,n} + \mathrm{r}_{i,n}] + F(t,t_{n+1})\mathrm{s}_{i,n}] \tag{19}$$

Where $G_{i,0}(t) = g_i(t)$

Since the function $u_{i,j}(t) = u_{i,j}$ is the approximate solution at (x_i, t_j) for equation (1).

3.2. The (BbyBM)

In this section, we use the **(BbyBM)** for solving the linear Fredholm-Volterra integral equation of the second kind. The interval $[\mathrm{a},\mathrm{b}]$ is divided into steps of width h, $t_j = jh$, $j = 0,1,...,n$ and $h = (b-a)/n$. the approximate solution of $u_i(t)$ will be define at mesh-points t_j and denoted by u_{ij}, $j = 0,1,...,n$ such as u_{ij} is an approximation to $u_i(t_j)$.

To solve the **(SLVIEs)**:

$$U(t) - \lambda\int_0^t F(t,\tau)U(\tau)d\tau = F(t) + \lambda\sum_{j=0}^{i} w_j k_{nj}U(t) \qquad ,n = 0,1,...N \tag{20}$$

Where,

$$
\begin{aligned}
&U(t) = (u_1(t), u_2(t), ..., u_l(t))^T, U(\tau) = (u_1(\tau), u_2(\tau), ..., u_l(\tau)), \\
&F(t) = (f_1(t), f_2(t), ..., f_l(t))^T
\end{aligned}
$$

And

$$F(t,\tau)U(\tau) = \begin{bmatrix} F_{1,1}(t,\tau)U(\tau) & F_{1,2}(t,\tau)U(\tau) & \dots & F_{1,s}(t,\tau)U(\tau) \\ F_{2,1}(t,\tau)U(\tau) & F_{2,2}(t,\tau)U(\tau) & \dots & F_{2,s}(t,\tau)U(\tau) \\ \vdots & \vdots & \ddots & \vdots \\ F_{s,1}(t,\tau)U(\tau) & F_{s,2}(t,\tau)U(\tau) & \dots & F_{s,s}(t,\tau)U(\tau) \end{bmatrix}$$

Then, we get

$$u_n(t)-\lambda\int_0^t F(t,\tau)U(\tau)d\tau = g_n(t) \qquad ,n=0,1,...,N, \tag{21}$$

Rewrite equation (21) as follows:

$$u_i(t_k)=g_i(t_k)+\lambda\int_0^{t_{pm}} F_{i,s}(t_k,\tau)U(\tau)d\tau+\lambda\int_{t_{pm}}^{t_n} F_{i,s}(t_k,\tau)U(\tau)d\tau$$

$$u_i(t_k)=f_i(t_k)+\lambda\sum_{n=0}^{i} w_n k_{in} u_n(t_k)+\lambda\int_0^{t_{pm}} F_{i,s}(t_k,\tau)U(\tau)d\tau+\lambda\int_{t_{pm}}^{t_n} F_{i,s}(t_k,\tau)U(\tau)d\tau \tag{22}$$

where $s=1,2,...$

If the values $u_{i0},u_{i1},...,u_{i,pm}$ are known, then the first integral can be approximated by standard quadrature methods, and the second integral is obtain by a quadrature rule using values of the integrand at $\tau=t_{pm},t_{pm+1},...,t_{p(m+1)}$.

Since the values of u_i at these points are unknown, we have a system of l_p linear equations by applied the Block-by-block method

$$\begin{aligned} u_{ik} &= f_i(t_k)+\lambda\sum_{n=0}^{i} w_n k_{in} u_n(t_k)+\lambda[h\sum_{j=0}^{mp} v_{kj} F_{i,s}(t_k,\tau_j)(u_{ij},...,u_{lj})] \\ &+\lambda[h\sum_{j=mp}^{(m+1)p} v'_{kj} F_{i,s}(t_k,\tau_{mp+j})(u_{1,mp+j},...,u_{l,mp+j})] \end{aligned} \tag{23}$$

For $n=mp+1,mp+2,...,(m+1)p$, $m=0,1,...,(N-1)$, where u_{kj},u^*_{kj} depend on the quadrature rule used.

Now, for the Modified method of two Blocks we take $p=2$,this integration over $[0,t_{2m}]$ can be accomplished by Simpson's rule, and the integral over $[t_{2m},t_n]$ by using a quadratic interpolation of the integrand at the point t_{2m},t_{2m+1},t_{2m+2}, then equation (21) becomes:

$$u_{i,2m+1}=g_i(t_{2m+1})+\lambda\int_0^{(2m+1)h} F_{i,s}(t_{2m+1},\tau)U(\tau)d\tau \tag{24}$$

And

$$u_{i,2m+2}=g_i(t_{2m+2})+\lambda\int_0^{(2m+2)h} F_{i,s}(t_{2m+2},\tau)U(\tau)d\tau \tag{25}$$

Where $i=1,2,...,l,\quad m=0,1,...$.

Therefore, by equation (23) the approximate solution is computed by:

$$u_{i,2m+1} = g_i(t_{2m+1}) + \lambda[\frac{h}{3}\sum_{j=0}^{2m} v_j F_{i,s}(t_{2m+1},\tau_j)(u_{1j},...,u_{lj}) \\ +\lambda[\frac{h}{12}[5F_{i,s}(t_{2m+1},\tau_{2m})(u_{1,2m},...,u_{l,2m}) + 8F_{i,s}(t_{2m+1},\tau_{2m+1}) \\ (u_{1,2m+1},...,u_{l,2m+1}) - F(t_{2m+1},\tau_{2m+2})(u_{1,2m+1},...,u_{l,2m+2})]] \tag{26}$$

$$u_{i,2m+2} = g_i(t_{2m+2}) + \lambda\frac{h}{3}\sum_{j=0}^{2m+2} v'_j F_{i,s}(t_{2m+2},\tau_j)(u_{1j},...,u_{lj}) \tag{27}$$

Where $u_{i,0} = g_i(t_0)$.

Thus, replace the second term in equation (26) by using integration formulas, then we get:

$$u_{i,2m+1} = g_i(t_{2m+1}) + \lambda[\frac{h}{3}\sum_{j=0}^{2m} v_j F_{i,s}(t_{2m+1},\tau_j)(u_{1j},...,u_{lj}) \\ +\lambda[\frac{h}{6}[F_{i,s}(t_{2m+1},\tau_{2m})(u_{1,2m},...,u_{l,2m}) + 4F_{i,s}(t_{2m+1},\tau_{2m+1/2}) \\ ((\frac{3}{8}u_{1,2m} + \frac{3}{4}u_{1,2m+1} - \frac{1}{8}u_{1,2m+2}),...,(\frac{3}{8}u_{l,2m} + \frac{3}{4}u_{l,2m+1} - \frac{1}{8}u_{l,2m+2})) \\ +F_{l,s}(t_{2m+1},\tau_{2m+1})(u_{1,2m+1},...,u_{l,2m+1})]] \tag{28}$$

$$u_{i,2m+2} = g_i(t_{2m+2}) + \lambda[\frac{h}{3}F_{i,s}(t_{2m+2},\tau_0)(u_{10},...,u_{l0}) + 4F_{i,s}(t_{2m+2},\tau_1)(u_{11},...,u_{l1}) \\ +...+F_{i,s}(t_{2m+2},\tau_{2m+2})(u_{1,2m+2},...,u_{l,2m+2})] \tag{29}$$

where

$$v_0 = v_{2m} = 1, \quad v_j = 3-(-1)^j, \quad j = 1,2,...,2m-1$$
$$v'_0 = v'_{2m+2} = 1, \quad v'_j = 3-(-1)^j, \quad j = 1,2,...,2m+1$$

Finally, we construct $2l$ linear equations from (28) and (29) to find the unknown functions $u_{i,2m+1}, u_{i,2m+2}$. The resulting system is solved by using modified Newton-Raphson method.

4. Numerical experiments and discussions

Example 1: Consider

$$u(x,t) = xt\,\frac{e^2+1}{4e+1} + \frac{5}{6}xt^3 - \int_0^1 xe^{2y-1}u(y,t)dy - \int_0^t (t+\tau)u(x,\tau)d\tau \tag{30}$$

where the exact solution is $u(x,t) = xt$ and $0 \le x,t \le 1$, here $\lambda = -1$, $\mu = 1$. In tables (4.1)-(4.4) we present the exact solution, the approximate numerical solutions by using **(R.KM)**and (**BbyBM),** and their corresponding errors for some points of (ix, it), $i = 0,1,...,N$,we suppose that $N = 10, 20$.

Case 1: $N = 10, \quad h = 0.1$

T	x	Exact sol.	$u^{R.K}$	$E^{R.K}$	$u^{B.B}$	$E^{B.B}$
0.2	0	0	0	0	0	0
	0.2	0.04000000	0.0398047279	1.952720×10^{-4}	0.0398568763	1.431236×10^{-4}
	0.4	0.08000000	0.0796094558	3.905441×10^{-4}	0.0797137526	2.862473×10^{-4}
	0.6	0.12000000	0.1194141835	5.858165×10^{-4}	0.1195706289	4.293711×10^{-4}
	0.8	0.16000000	0.1592189115	7.810885×10^{-4}	0.1594275054	5.724946×10^{-4}
	1.0	0.20000000	0.1990236395	9.763605×10^{-4}	0.1992843816	7.156184×10^{-4}
0.6	0	0	0	0	0	0
	0.2	0.12000000	0.1183707219	1.629278×10^{-3}	0.1196300027	3.699973×10^{-4}
	0.4	0.24000000	0.2367414437	3.258556×10^{-3}	0.2392600050	7.399950×10^{-4}
	0.6	0.36000000	0.3551121654	4.887834×10^{-3}	0.3588900081	1.109991×10^{-3}
	0.8	0.48000000	0.4734828877	6.517112×10^{-3}	0.4785200113	1.479988×10^{-3}
	1.0	0.60000000	0.5918536075	8.146392×10^{-3}	0.5981500141	1.849985×10^{-3}
1.0	0	0	0	0	0	0
	0.2	0.20000000	0.1948114702	5.188529×10^{-3}	0.1995393562	4.606438×10^{-4}
	0.4	0.40000000	0.3896229407	1.037705×10^{-2}	0.3990787120	9.212880×10^{-4}
	0.6	0.60000000	0.5844344110	1.556558×10^{-2}	0.5986180678	1.381932×10^{-3}
	0.8	0.80000000	0.7792458806	2.075411×10^{-2}	0.7981574246	1.842575×10^{-3}
	1.0	1.00000000	0.9740573519	2.594264×10^{-2}	0.9976967818	2.303218×10^{-3}

Table 4.1

Case 2: $N = 20, \quad h = 0.05$

T	x	Exact sol.	$u^{R.K}$	$E^{R.K}$	$u^{B.B}$	$E^{B.B}$
0.2	0	0	0	0	0	0
	0.2	0.04000000	0.0399381695	6.183050×10^{-5}	0.0399640942	3.590580×10^{-5}
	0.4	0.08000000	0.0798763390	1.236609×10^{-4}	0.0799281883	7.181168×10^{-5}
	0.6	0.12000000	0.1198145093	1.854907×10^{-4}	0.1198922824	1.077176×10^{-4}
	0.8	0.16000000	0.1597526788	2.473212×10^{-4}	0.1598563765	1.436235×10^{-4}
	1.0	0.20000000	0.1996908472	3.091528×10^{-4}	0.1998204709	1.795291×10^{-4}
0.6	0	0	0	0	0	0
	0.2	0.12000000	0.1192787564	7.212436×10^{-4}	0.1199072153	9.27847×10^{-5}
	0.4	0.24000000	0.2385575133	1.442486×10^{-3}	0.2398144307	1.855693×10^{-4}
	0.6	0.36000000	0.3578362700	2.163730×10^{-3}	0.3597216462	2.783538×10^{-4}
	0.8	0.48000000	0.4771150266	2.884973×10^{-3}	0.4796288617	3.711383×10^{-4}
	1.0	0.60000000	0.5963937818	3.606218×10^{-3}	0.5995360768	4.639232×10^{-4}
1.0	0	0	0	0	0	0
	0.2	0.20000000	0.1975122164	2.487783×10^{-3}	0.1998845727	1.154273×10^{-4}
	0.4	0.40000000	0.3950244325	4.975567×10^{-3}	0.3997691459	2.308541×10^{-4}
	0.6	0.60000000	0.5925366498	7.463350×10^{-3}	0.5996537181	3.462819×10^{-4}
	0.8	0.80000000	0.7900488614	9.951138×10^{-3}	0.7995382913	4.617087×10^{-4}
	1.0	1.00000000	0.9875610805	1.24389×10^{-2}	0.9994228675	5.771325×10^{-4}

Table(4.2)

Example 2: Consider:

$$u(x,t) = \left(\frac{4}{3} + \frac{3t}{2}\right)x + t + t\sin(t) - (tx + t^2)\cos(t) - \int_0^1 xyu(y,t)dy - \int_0^t t\sin(s)u(x,s)ds \qquad (31)$$

where the exact solution is $u(x,t) = x + t$ and $0 \le x, t \le 1$, here $\lambda = -1,\ \mu = 1$.

Case 1: $N = 10, \quad h = 0.1$.

T	x	Exact sol.	$u^{R.K}$	$E^{R.K}$	$u^{B.B}$	$E^{B.B}$
0.2	0	0.20000000	0.2000000031	3.1 X 10^{-8}	0.2000000002	2.0× 10^{-10}
	0.2	0.40000000	0.3997479808	2.520192 X 10^{-4}	0.3997510573	2.489427 X 10^{-4}
	0.4	0.60000000	0.5994959587	5.040413 X 10^{-4}	0.5995021139	4.978861 X 10^{-4}
	0.6	0.80000000	0.7992439356	7.560644 X 10^{-4}	0.7992531713	7.468287 X 10^{-4}
	0.8	1.00000000	0.9989919165	1.008083 X 10^{-3}	0.9990042290	9.957710 X 10^{-4}
	1.0	1.20000000	1.198739889	1.260111 X 10^{-3}	1.198755280	1.244720 X 10^{-3}
0.6	0	0.60000000	0.6000054041	5.4041 X 10^{-6}	0.6000000898	8.98 X 10^{-8}
	0.2	0.80000000	0.7995406302	4.593698 X 10^{-4}	0.7997693941	2.306059 X 10^{-4}
	0.4	1.00000000	0.9990758539	9.241461 X 10^{-4}	0.9995386990	4.613010 X 10^{-4}
	0.6	1.20000000	1.198611078	1.388922 X 10^{-3}	1.199308001	6.919990 X 10^{-4}
	0.8	1.40000000	1.398146307	1.853693 X 10^{-3}	1.399077305	9.226950 X 10^{-4}
	1.0	1.60000000	1.597681534	2.318466 X 10^{-3}	1.598846611	1.153389 X 10^{-3}
1.0	0	1.00000000	1.000125541	1.255410 X 10^{-4}	0.9999978454	1.360000 X 10^{-6}
	0.2	1.20000000	1.198419766	1.580234 X 10^{-3}	1.200958179	1.732260 X 10^{-4}
	0.4	1.40000000	1.396713994	3.286006 X 10^{-3}	1.401918512	3.478190 X 10^{-4}
	0.6	1.60000000	1.595008222	4.991778 X 10^{-3}	1.602878847	5.224030 X 10^{-4}
	0.8	1.80000000	1.793302449	6.697551 X 10^{-3}	1.803839184	6.969940 X 10^{-4}
	1.0	2.00000000	1.991596676	8.403324 X 10^{-2}	2.004799519	8.715820 X 10^{-4}

Table(4.3)

Case 2: $N = 20, \quad h = 0.05$

T	x	Exact sol.	$u^{R.K}$	$E^{R.K}$	$u^{B.B}$	$E^{B.B}$
0.2	0	0.20000000	0.2000000008	8.0000 X 10^{-8}	0.2000000001	1.00000 X 10^{-9}
	0.2	0.40000000	0.3999361903	6.38097 X 10^{-5}	0.3999377059	6.229410 X 10^{-5}
	0.4	0.60000000	0.5998723793	1.276207 X 10^{-4}	0.5998754120	1.245880 X 10^{-4}
	0.6	0.80000000	0.7998085710	1.914290 X 10^{-4}	0.7998131152	1.868848 X 10^{-4}
	0.8	1.00000000	0.9997447569	2.552431 X 10^{-4}	0.9997508208	2.491792 X 10^{-4}
	1.0	1.20000000	1.199680948	3.190520 X 10^{-4}	1.1996885370	3.114630 X 10^{-4}
0.6	0	0.60000000	0.6000012764	1.276400 X 10^{-6}	0.6000000054	5.400000 X 10^{-9}
	0.2	0.80000000	0.7998278328	1.721672 X 10^{-4}	0.7999422834	5.771660 X 10^{-5}
	0.4	1.00000000	0.999654390	3.456092 X 10^{-4}	0.9998845579	1.154421 X 10^{-4}
	0.6	1.20000000	1.199480945	5.190550 X 10^{-4}	1.199826839	1.731610 X 10^{-4}
	0.8	1.40000000	1.399307497	6.925030 X 10^{-4}	1.399769113	2.308870 X 10^{-4}
	1.0	1.60000000	1.599134056	8.659440 X 10^{-4}	1.599711396	2.886040 X 10^{-4}
1.0	0	1.00000000	1.000029747	2.974700 X 10^{-5}	1.000000085	8.500000 X 10^{-8}
	0.2	1.20000000	1.199225760	7.742400 X 10^{-4}	1.199956434	4.356600 X 10^{-5}
	0.4	1.40000000	1.398421769	1.578231 X 10^{-3}	1.399912786	8.721400 X 10^{-5}
	0.6	1.60000000	1.597617769	2.382231 X 10^{-3}	1.599869132	1.308680 X 10^{-4}
	0.8	1.80000000	1.796813772	3.186228 X 10^{-3}	1.799825483	1.745170 X 10^{-4}
	1.0	2.00000000	1.996009788	3.990212 X 10^{-3}	1.999781833	2.181670 X 10^{-4}

Table(4.4)

5. The Conclusion

From the previous discussions we conclude the following:

1. As x and t are increasing in $[0,1]\times[0,1]$, the errors due to **(R.KM)** and **(BbyBM)** are also increasing .
2. As N increasing ,the error also decreasing.
3. The errors due to the **(BbyBM)** less than the errors due to the **(R.KM)** (i.e. **(BbyBM)** better than **(R.KM)**).

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