STATISTICAL ANALYSIS FOR IMPRECISE TIMES TO BREAKDOWN DATASET USING THE INVERSE WEIBULL DISTRIBUTION

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ABSTRACT

One of the common data assumptions in statistics is preciseness and accuracy of data. However, in real world situations, machine or human errors or some unexpected situations can cause data to be imprecise. In this case fuzzy logic besides the classical statistical techniques should be employed. This paper is interested in statistically analyzing times to breakdown dataset using the inverse Weibull distribution when the assumption of the preciseness of the data is violated. Therefore first this study has found estimates of the parameters of the inverse Weibull distribution in presence of imprecise data. Both the classical maximum likelihood estimation and the Bayesian estimation have been utilized to estimate the parameters of the inverse Weibull in presence of imprecise data. Both the Newton-Raphson and expectation maximization algorithms have been utilized to find the maximum likelihood estimates. Second, the proposed methods have been utilized to find estimates of the parameters of the inverse Weibull distribution for the times to breakdown dataset. The estimates attained using the three methods are close to each other.

Keywords: *Statistical Analysis, Inverse Weibull distribution, Maximum likelihood estimation, Bayesian estimation, Imprecise data.*

1. INTRODUCTION

In 1939, the Swedish physicist Waloddi Weibull developed the Weibull distribution to describe the breaking strength of materials (Zanakis [17]). Since then, the Weibull distribution has been used in different fields such as engineering, business, hydrology, biology, forestry and of course, statistics. Similarly, the inverse Weibull (IW) distribution can be applied in different fields such as medicine branching process, ecology and biological studies (Sherina and Oluyede [11]). Properties of the inverse Weibull distribution are given by Johnson et al. [6], Keller [7], Calabria and Pulcini [1-3] and Khan et al. [8].

A random variable X is said to have the inverse Weibull distribution if the corresponding probability density function and cumulative distribution function are given, respectively, by

$$f(x;\ \alpha,\lambda) = \beta\alpha^{\beta}x^{-(\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}, \quad \alpha > 0, \beta > 0, x > 0, \tag{1.1}$$

$$F(x;\ \alpha,\lambda) = e^{-(\frac{\alpha}{x})^{\beta}}, \tag{1.2}$$

The fuzzy, as a word, means distorted, indistinct or unclear. Other meanings are not clearly expressed or thought out. Computations in fuzzy logic do not depend on the usual true or false which utilized in Boolean logic. It depends on the “the degree of truth” or the membership function which takes any value from zero to 100% rather than zero or 100% as in the Boolean logic. Therefore, fuzzy logic can be viewed as a generalization of Boolean logic. Recently, several articles adopted the fuzzy approach and worked on generalizing classical statistical methods such that it can handle fuzzy or imprecise data. For example, Viertl [13] presented generalized classical statistical inference methods for univariate fuzzy data. Wu [15] developed Bayesian estimation under fuzzy environments for lifetime data. Pak [9], Pak and Mahmoudi [10] conducted statistical inference for lifetime distributions based on fuzzy data.

According to Viertl [14] data measurements of continuous data such as time, volume and length are always imprecise real numbers. Hence, the aim of this paper is to provide statistical inference for the breaking down dataset taking in consideration the dataset is not precise utilizing the inverse Weibull distribution which was fitted before to the same dataset but with the assumption that the dataset is precise (see Erto [5]). First, the estimates of the parameters of the inverse Weibull distribution for fuzzy data are found using the Maximum likelihood and the Bayesian estimation where two algorithms are utilized for finding the maximum likelihood estimates. These algorithms are the Newton Raphson and the expectation maximization algorithm. Second, the times to breakdown dataset was fitted to the inverse Weibull distribution using the three proposed methods. The rest of this paper is organized as follows. Section 2 presents the Maximum Likelihood estimation for the parameters of the inverse Weibull distribution when the data is imprecise using two algorithms while Section 3 gives the Bayesian estimation. Methods of estimation presented in Section 2 and 3 applied to the times to breakdown dataset in Section 4.

2. MAXIMUM LIKELIHOOD ESTIMATION

Let $X_1, X_2, \dots, X_n$ be a random sample of size n drawn from the inverse Weibull distribution with probability density function given by (1.1). Assume that these variables are independent and identically distributed and the information about the measurements of these experimental units is received imprecisely. The partial information about $\boldsymbol{x}$ is assumed to be available in the form of fuzzy observation $\tilde{\boldsymbol{x}} = \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ with the Borel measurable membership function $\mu_{\tilde{x}}(\mathbf{x})$. Zadeh [16] defined the probability of a fuzzy event which can be employed to define the likelihood function as

$$L(\alpha, \beta; \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) = \prod_{i=1}^{n} \int \beta\alpha^{\beta} x^{-(\beta+1)} e^{-(\frac{\alpha}{x})^{\beta}} \mu_{\tilde{x}_i}(x)\, dx$$

$$= \beta^{n} \alpha^{n\beta} \prod_{i=1}^{n} \int x^{-(\beta+1)} e^{-(\frac{\alpha}{x})^{\beta}} \mu_{\tilde{x}_i}(x)\, dx \qquad (2.1)$$

The corresponding log-likelihood function $L^*(\alpha, \beta) = \log\left(L(\alpha, \beta; \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)\right)$ is

$$L^*(\alpha, \beta) = n\log(\beta) + (n\beta)\ \log(\alpha) + \textstyle\sum_{i=1}^{n} log \int x^{-(\beta+1)} e^{-(\frac{\alpha}{x})^{\beta}} \mu_{\tilde{x}_i}(x)\, dx \qquad (2.2)$$

Maximizing the observed-data log-likelihood (2.2), the maximum likelihood estimate of the parameter can be achieved. Equating the derivative of the log-likelihood with respect to zero, one has

$$\frac{\partial}{\partial\alpha} L^*(\alpha, \beta) = \frac{n\beta}{\alpha} - \sum_{i=1}^{n} \frac{\int \beta\, e^{-(\frac{\alpha}{x})^{\beta}} x^{-2-\beta} (\frac{\alpha}{x})^{-1+\beta} \mu_{\tilde{x}_i}(\mathrm{x}) dx}{\int x^{-(\beta+1)} e^{-(\frac{\alpha}{x})^{\beta}} \mu_{\tilde{x}_i}(x)\, dx} = 0 \qquad (2.3)$$

$$\frac{\partial}{\partial\beta} L^*(\alpha, \lambda) = \frac{n}{\beta} + n\ \log(\alpha) - \sum_{i=1}^{n} \frac{\int e^{-\left(\frac{\alpha}{x}\right)^{\beta}} x^{-1-\beta} \left(\mathrm{Log}[x] + \left(\frac{\alpha}{x}\right)^{\beta} \mathrm{Log}\left[\frac{\alpha}{x}\right]\right) \mu_{\tilde{x}_i}(\mathrm{x}) dx}{\int x^{-(\beta+1)} e^{-\left(\frac{\alpha}{x}\right)^{\beta}} \mu_{\tilde{x}_i}(x) dx} = 0. \quad (2.4)$$

An iterative numerical algorithm should be used to obtain the maximum likelihood estimates because there are no closed form of the solutions to the likelihood equations Equation (2.3) and (2.4). Hence this study describes the Newton-Raphson method and the expectation maximization algorithm to determine the maximum likelihood estimates of α and β.

2.1 Newton-Raphson Algorithm

In the Newton-Raphson algorithm, the likelihood equation can be solved as a result of an iterative procedure. Let $\boldsymbol{\gamma} = (\alpha, \lambda)^{T}$ be the parameter vector, then in the (h+1)th iteration, the updated parameter can be attained as

$$\boldsymbol{\gamma}^{(h+1)} = \boldsymbol{\gamma}^{(h)} - \left[\frac{\partial^2 L^*(\boldsymbol{\gamma}; \tilde{\boldsymbol{x}})}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}^{T}} \Big| \boldsymbol{\gamma} = \boldsymbol{\gamma}^{(h)}\right]^{-1} \left[\frac{\partial L^*(\boldsymbol{\gamma}; \tilde{\boldsymbol{x}})}{\partial\boldsymbol{\gamma}} \Big| \boldsymbol{\gamma} = \boldsymbol{\gamma}^{(h)}\right] \qquad (2.5)$$

where

$$\frac{\partial L^*(\boldsymbol{\gamma}; \tilde{\boldsymbol{x}})}{\partial\boldsymbol{\gamma}} = \begin{pmatrix} \frac{\partial L^*(\alpha, \lambda; \tilde{\boldsymbol{x}})}{\partial\alpha} \\ \frac{\partial L^*(\alpha, \lambda; \tilde{\boldsymbol{x}})}{\partial\beta} \end{pmatrix},$$

and

$$\frac{\partial^2 L^*(\boldsymbol{\gamma};\widetilde{\boldsymbol{x}})}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}^T}=\begin{pmatrix}\frac{\partial^2 L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\alpha^2} & \frac{\partial^2 L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\alpha\partial\beta}\\ \frac{\partial^2 L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\alpha\partial\beta} & \frac{\partial^2 L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\beta^2}\end{pmatrix}$$

The following are the second-order derivatives of the log-likelihood with respect to the parameter required for proceeding with the Newton-Raphson algorithm:

$$\frac{\partial^2 L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\alpha^2}=-\frac{n\beta}{\alpha^2}+\sum_{i=1}^{n}\left(\frac{\int\alpha^{-3}e^{-(\frac{\alpha}{x})^{\beta}}x^{-\beta}(\frac{\alpha}{x})^{1+\beta}\beta(1+(-1+(\frac{\alpha}{x})^{\beta})\beta)\,\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int t^{-(\beta+1)}e^{-(\frac{\alpha}{t})^{\beta}}\mu_{\tilde{x}_i}(x)\,dx}-\left[\frac{\int\beta e^{-(\frac{\alpha}{x})^{\beta}}x^{-2-\beta}(\frac{\alpha}{x})^{\beta-1}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int x^{-(\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}\mu_{\tilde{x}_i}(x)\,dx}\right]^2\right)$$

$$\frac{\partial^2 L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\beta^2}=-\frac{n}{\beta^2}+\sum_{i=1}^{n}\left(\frac{-e^{-(\frac{\alpha}{x})^{\beta}}x^{-1-\beta}(-\mathrm{Log}[x]^2-2(\frac{\alpha}{x})^{\beta}\mathrm{Log}[x]\mathrm{Log}[\frac{\alpha}{x}]-(\frac{\alpha}{x})^{\beta}(-1+(\frac{\alpha}{x})^{\beta})\mathrm{Log}[\frac{\alpha}{x}]^2)\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int t^{-(\beta+1)}e^{-(\frac{\alpha}{t})^{\beta}}\mu_{\tilde{x}_i}(x)\,dx}-\left[\frac{\int e^{-(\frac{\alpha}{x})^{\beta}}x^{-1-\beta}(\mathrm{Log}[x]+(\frac{\alpha}{x})^{\beta}\mathrm{Log}[\frac{\alpha}{x}])\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int x^{-(\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}\mu_{\tilde{x}_i}(x)\,dx}\right]^2\right)$$

$$\frac{\partial^2 L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\alpha\partial\lambda}=\sum_{i=1}^{n}\frac{\int e^{-(\frac{\alpha}{x})^{\beta}}t^{-2-\beta}(\frac{\alpha}{x})^{-1+\beta}(-1+\beta\mathrm{Log}[x]+(-1+(\frac{\alpha}{x})^{\beta})\beta\mathrm{Log}[\frac{\alpha}{x}])\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int x^{-(\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}\mu_{\tilde{x}_i}(x)\,dx}+\sum_{i=1}^{n}\left(\frac{\int\beta e^{-(\frac{\alpha}{x})^{\beta}}\frac{\alpha^{-2-\beta}}{x}(\frac{\alpha}{x})^{-1+\beta}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int x^{-(\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}\mu_{\tilde{x}_i}(x)\,dx}\times\frac{\int e^{-(\frac{\alpha}{x})^{\beta}}x^{-1-\beta}(\mathrm{Log}[x]+(\frac{\alpha}{x})^{\beta}\mathrm{Log}[\frac{\alpha}{x}])\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int x^{-(\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}\mu_{\tilde{x}_i}(x)\,dx}\right)$$

The interation process should proceed until convergence is achieved, i.e., until $\left\|\boldsymbol{\gamma}^{(h+1)}-\boldsymbol{\gamma}^{(h)}\right\|<\varepsilon$, for some prefixed $\varepsilon>0$.

The next subsection discusses the expectation maximization algorithm since it is viable alternative to the Newton-Raphson algorithm.

2.2 EM Algorithm

The expectation maximization algorithm is an extensively applicable approach to iterative computation of maximum likelihood estimates in case of incomplete-data problems. The observed fuzzy data $\widetilde{\boldsymbol{x}}$ is an incomplete specification of a complete vector $\boldsymbol{x}$. The fuzzy expectation maximization algorithm (see Denoeux [4]) is employed to determine the maximum likelihood estimates of α and β

For a complete known realization $\boldsymbol{x}=(x_1,x_2,\dots,x_n)$ of $\boldsymbol{X}$, the complete-data likelihood function is

$$l_0(\alpha,\beta;\boldsymbol{x})=\beta^n\alpha^{n\beta}e^{-\alpha^{\beta}\sum_{i=1}^{n}x_i{}^{-\beta}}\prod_{i=1}^{n}x_i{}^{-(\beta+1)} \tag{2.6}$$

From equation (2.6), the log-likelihood function for the complete data vector $\boldsymbol{x}$ is

$$\log l_0\,(\alpha,\beta;\boldsymbol{x}) = n\log(\beta) + n\,\beta\log(\alpha) - \alpha^{\beta}\sum_{i=1}^{n} x_i^{-\beta} - (\beta+1)\sum_{i=1}^{n}\log(x_i) \qquad (2.7)$$

The following equations are the results of taking the derivatives with respective to α and β, respectively, on (2.7):

$$\frac{n}{\alpha^{\beta}} = \sum_{i=1}^{n} x_i^{-\beta} \qquad (2.8)$$

$$\frac{n}{\beta} = -\mathrm{n}\ \log(\alpha) + \alpha^{\beta}\left(\mathrm{Log}[\alpha]\sum_{i=1}^{n} x_i^{-\beta} - \sum_{i=1}^{n}\mathrm{Log}[x_i]x_i^{-\beta}\right) + \sum_{i=1}^{n}\log(x_i) \qquad (2.9)$$

The following steps should be performed for the expectation maximization algorithm:

1. set $h = 0$ and choose starting values of α and β, say $\alpha^{(0)}$ and $\beta^{(0)}$
2. The following two steps should be performed in each (h+1)th iteration:
   - First Step, The E-Step: in this steps one should find the following conditional expectations:

$$E_{1i} = E_{\alpha^{(h)},\lambda^{(h)}}\left(t^{-\beta}\middle|\tilde{x}_i\right) = \frac{\int t^{-(2\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}\mu_{\tilde{x}_i}(x)\,dx}{\int t^{-(\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}\mu_{\tilde{x}_i}(x)\,dx}$$

$$E_{2i} = E_{\alpha^{(h)},\lambda^{(h)}}(\log(t)|\tilde{x}_i) = \frac{\int \log(t)\,t^{-(\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}\mu_{\tilde{x}_i}(x)\,dx}{\int t^{-(\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}\mu_{\tilde{x}_i}(x)\,dx}$$

$$E_{3i} = E_{\alpha^{(h)},\lambda^{(h)}}\left(\mathrm{Log[t]}t^{-\beta}\middle|\tilde{x}_i\right) = \frac{\int \mathrm{Log}(t)\,t^{-(2\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}\mu_{\tilde{x}_i}(x)\,dx}{\int t^{-(\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}\mu_{\tilde{x}_i}(x)\,dx}$$

Then, equations (2.8) and (2.9) should be replaced by

$$\beta^{(h+1)} = \frac{\mathrm{n}}{\sum_{i=1}^{n}E_{2i} - (\sum_{i=1}^{n}E_{1i})(\sum_{i=1}^{n}E_{3i})/n} \qquad (2.10)$$

$$\alpha^{(h+1)} = \left(\frac{n}{\sum_{i=1}^{n}E_{1i}}\right)^{1/\beta} \qquad (2.11)$$

   - The Second Step, The M-Step: In this step one should solve the equations (2.10) and (2.11) and obtain the next values, $\alpha^{(h+1)}$ and $\lambda^{(h+1)}$, of α and λ, respectively as follows:

3. If convergence occurs then $\alpha^{(h+1)}$ and $\lambda^{(h+1)}$ are the maximum likelihood estimates of α and λ via the expectation maximization; otherwise, set $h = h + 1$ and go to step 2.

3. BAYESIAN ESTIMATION

Under the Bayesian estimation setting the unknown parameter is assumed to be a random variable with distribution usually known as prior probability distribution. In this paper, the conjugate prior for α is assumed to be Gamma(a, b) density of the form:

$$\pi_1(\alpha) \propto \alpha^{a-1} e^{-\alpha b}, \qquad \alpha > 0, \tag{3.1}$$

and for the parameter β it is assumed to be Gamma(c, d) density of the form:

$$\pi_2(\beta) \propto \beta^{c-1} e^{-\beta d}, \qquad \lambda > 0, \tag{3.2}$$

where $a > 0$, $b > 0$, $c > 0$ and $d > 0$. Under these priors, and given the data, the joint posterior density function of α and β can be obtained as

$$\pi(\alpha, \beta|\tilde{\boldsymbol{x}}) = \frac{\pi_1(\alpha)\, \pi_2(\beta) \ell(\alpha, \beta; \tilde{\boldsymbol{x}})}{\int_0^\infty \int_0^\infty \pi_1(\alpha)\, \pi_2(\beta) \ell(\alpha, \beta; \tilde{\boldsymbol{x}})\, d\alpha d\beta} \tag{3.3}$$

where

$$\ell(\alpha, \beta; \tilde{\boldsymbol{x}}) = \beta^{n+c-1} \alpha^{n\beta+a-1} e^{-\alpha b} e^{-\beta d} \prod_{i=1}^{n} \int x^{-(\beta+1)} e^{-(\frac{\alpha}{x})^\beta} \mu_{\tilde{x}_i}(x)\, dx$$

is the likelihood function for the imprecise sample $\tilde{\boldsymbol{x}}$. The Bayes estimate of any function of α and λ, say $g(\alpha, \lambda)$, under the squared error loss function is given by

$$E(g(\alpha, \beta)|\tilde{\boldsymbol{x}}) = \frac{\int_0^\infty \int_0^\infty g(\alpha, \beta) \pi_1(\alpha)\, \pi_2(\beta) \ell(\alpha, \beta; \tilde{\boldsymbol{x}})\, d\alpha d\beta}{\int_0^\infty \int_0^\infty \pi_1(\alpha)\, \pi_2(\beta) \ell(\alpha, \beta; \tilde{\boldsymbol{x}})\, d\alpha d\beta}$$

$$= \frac{\int_0^\infty \int_0^\infty g(\alpha, \beta) e^{Q(\alpha,\beta)} d\alpha d\beta}{\int_0^\infty \int_0^\infty e^{Q(\alpha,\beta)}\, d\alpha d\beta} \tag{3.4}$$

where $Q(\alpha, \beta) = \ln[\pi_1(\alpha)\pi_2(\beta)] + \ln \ell(\alpha, \beta; \tilde{\boldsymbol{x}}) \equiv \rho(\alpha, \beta) + L(\alpha, \beta)$. The expression in (3.4) can be reexpressed as

$$E(g(\alpha, \beta)|\tilde{\boldsymbol{x}}) == \frac{\int_0^\infty \int_0^\infty e^{nH^*(\alpha,\lambda)} d\alpha d\beta}{\int_0^\infty \int_0^\infty e^{nH^*(\alpha,\lambda)}\, d\alpha d\beta} \tag{3.5}$$

where

$$H(\alpha, \beta) = \frac{1}{n}\, Q(\alpha, \beta),$$

and

$$H^*(\alpha, \beta) = \frac{1}{n} [\ln g(\alpha, \beta) + Q(\alpha, \beta)]$$

An approximation of (3.5) can be attained by Tierney and Kadane [12] using Laplace's method as follows:

$$\hat{g}_{BT}(\theta) = \left[\frac{\det \Sigma^*}{\det \Sigma}\right]^{1/2} e^{n\,[H^*(\bar{\alpha}^*, \bar{\beta}^*) - H(\bar{\alpha}, \bar{\beta})]}, \tag{3.6}$$

where $(\bar{\alpha}^*, \bar{\beta}^*)$ and $(\bar{\alpha}, \bar{\beta})$ maximize $H^*(\alpha, \beta)$ and $H(\alpha, \beta)$, respectively. The negative of the inverse Hessians of $H(\alpha, \beta)$ and $H^*(\alpha, \beta)$ at $(\bar{\alpha}, \bar{\beta})$ and $(\bar{\alpha}^*, \bar{\beta}^*)$ are Σ and Σ^*, respectively.

To apply this approximation to obtain the Bayes estimate of the parameter α and β, one has

$$H(\alpha, \beta) = \frac{1}{n}\{k + (n + c - 1) \log(\beta) + (n\beta + a - 1) \log(\alpha)$$

$$-\alpha b-\beta d+\sum_{i=1}^{n} log\int x^{-(\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}\mu_{\tilde{x}_i}(\mathrm{x})dx\}$$

where k is a constant. Solving the following two equations yields $(\bar{\alpha},\bar{\beta})$:

$$\frac{\partial}{\partial\alpha}H(\alpha,\beta)=\frac{1}{n}\left\{\frac{(n\beta+a-1)}{\alpha}-b-\sum_{i=1}^{n}\frac{\int\beta\, e^{-(\frac{\alpha}{x})^{\beta}}x^{-2-\beta}(\frac{\alpha}{x})^{-1+\beta}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int x^{-(\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\right\}$$

$$\frac{\partial}{\partial\beta}H(\alpha,\beta)=\frac{1}{n}\left\{\frac{(n+c-1)}{\beta}+n\ \log(\alpha)-d-\sum_{i=1}^{n}\frac{\int e^{-(\frac{\alpha}{x})^{\beta}}x^{-1-\beta}(\mathrm{Log}[x]+(\frac{\alpha}{x})^{\beta}\mathrm{Log}[\frac{\alpha}{x}])\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int x^{-(\beta+1)}e^{-(\frac{\alpha}{x})^{\beta}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\right\}$$

The determinant of the negative of the inverse Hessian $H(\alpha,\beta)$ at $(\bar{\alpha},\bar{\beta})$ can be attained using the second derivative of $H(\alpha,\beta)$ which is given by:

$$\det\Sigma=(H_{11}H_{22}-H_{12}^{2})^{-1}$$

where

$$H_{11}=\frac{1}{n}\left\{-\frac{(n\bar{\beta}+a-1)}{\bar{\alpha}^{2}}+\sum_{i=1}^{n}\left(\frac{\int\bar{\alpha}^{-3}e^{-(\frac{\alpha}{x})^{\beta}}x^{-\beta}(\frac{\bar{\alpha}}{x})^{1+\beta}\beta(1+(-1+(\frac{\bar{\alpha}}{x})^{\bar{\beta}})\bar{\beta})\,\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int x^{-(\bar{\beta}+1)}e^{-(\frac{\bar{\alpha}}{x})^{\bar{\beta}}}\mu_{\tilde{x}_i}(x)\,dx}-\left[\frac{\int\bar{\beta}e^{-(\frac{\bar{\alpha}}{x})^{\bar{\beta}}}x^{-2-\bar{\beta}}(\frac{\bar{\alpha}}{x})^{\bar{\beta}-1}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int x^{-(\bar{\beta}+1)}e^{-(\frac{\bar{\alpha}}{x})^{\bar{\beta}}}\mu_{\tilde{x}_i}(x)\,dx}\right]^{2}\right)\right\}$$

$$H_{22}=\frac{1}{n}\left\{-\frac{(n+c-1)}{\bar{\beta}^{2}}+\sum_{i=1}^{n}\left(\frac{-e^{-(\frac{\bar{\alpha}}{x})^{\bar{\beta}}}x^{-1-\bar{\beta}}(-\mathrm{Log}[x]^{2}-2(\frac{\bar{\alpha}}{x})^{\bar{\beta}}\mathrm{Log}[t]\mathrm{Log}[\frac{\bar{\alpha}}{x}]-(\frac{\bar{\alpha}}{x})^{\bar{\beta}}(-1+(\frac{\bar{\alpha}}{x})^{\bar{\beta}})\mathrm{Log}[\frac{\bar{\alpha}}{x}]^{2})\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int x^{-(\bar{\beta}+1)}e^{-(\frac{\bar{\alpha}}{x})^{\bar{\beta}}}\mu_{\tilde{x}_i}(x)\,dx}-\left[\frac{\int e^{-(\frac{\bar{\alpha}}{x})^{\bar{\beta}}}x^{-1-\bar{\beta}}(\mathrm{Log}[x]+(\frac{\bar{\alpha}}{x})^{\bar{\beta}}\mathrm{Log}[\frac{\bar{\alpha}}{x}])\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int x^{-(\bar{\beta}+1)}e^{-(\frac{\bar{\alpha}}{x})^{\bar{\beta}}}\mu_{\tilde{x}_i}(x)\,dx}\right]^{2}\right)\right\}$$

$$H_{12}=\frac{1}{n}\sum_{i=1}^{n}\frac{\int e^{-(\frac{\bar{\alpha}}{x})^{\bar{\beta}}}x^{-2-\bar{\beta}}(\frac{\bar{\alpha}}{x})^{-1+\bar{\beta}}(-1+\bar{\beta}\mathrm{Log}[x]+(-1+(\frac{\bar{\alpha}}{x})^{\bar{\beta}})\bar{\beta}\mathrm{Log}[\frac{\bar{\alpha}}{x}])\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int x^{-(\bar{\beta}+1)}e^{-(\frac{\bar{\alpha}}{x})^{\bar{\beta}}}\mu_{\tilde{x}_i}(x)\,dx}+\sum_{i=1}^{n}\left(\frac{\int\bar{\beta}e^{-(\frac{\bar{\alpha}}{x})^{\bar{\beta}}}x^{-2-\bar{\beta}}(\frac{\bar{\alpha}}{x})^{-1+\bar{\beta}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int x^{-(\bar{\beta}+1)}e^{-(\frac{\bar{\alpha}}{x})^{\bar{\beta}}}\mu_{\tilde{x}_i}(x)\,dx}\times\frac{\int e^{-(\frac{\bar{\alpha}}{x})^{\bar{\beta}}}x^{-1-\bar{\beta}}(\mathrm{Log}[x]+(\frac{\bar{\alpha}}{x})^{\bar{\beta}}\mathrm{Log}[\frac{\bar{\alpha}}{x}])\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int x^{-(\bar{\beta}+1)}e^{-(\frac{\bar{\alpha}}{x})^{\bar{\beta}}}\mu_{\tilde{x}_i}(x)\,dx}\right)\Bigg\}$$

Following the same arguments with $g(\alpha,\beta)=\alpha$ and β, respectively, in $H^*(\alpha,\beta)$, $\hat{\alpha}_{BT},\hat{\beta}_{BT}$ in equation (3.6) can be attained directly.

4. APPLICATION

In this Section, the proposed methods in Section 2 and 3 are applied on times to breakdown of a Capacitor Insulating Fluid dataset which is presented in Table 1. The dataset is an instance of the critical real-world situations in which only small datasets are available. This data set was fitted to the inverse Weibull distribution by Erto [5]. Here the same approach is followed but with taking in consideration that the data measurements are imprecise since it is continuous quantity. One may assume that the imprecision in the times to breakdown dataset can be represented by the following fuzzy information system:

$$\mu_{\tilde{x}_1}(x)=\begin{cases}1 & x\le 0.10,\\ \dfrac{0.40-x}{0.30} & .10\le x\le 0.40,\\ 0 & otherwise,\end{cases}$$

$$\mu_{\tilde{x}_2}(x)=\begin{cases}\frac{x-0.10}{0.30} & .10\le x\le 0.40,\\ \frac{0.9-x}{0.50} & .40\le x\le 0.9,\\ 0 & otherwise,\end{cases}$$

$$\mu_{\tilde{x}_3}(x)=\begin{cases}\dfrac{x-0.40}{0.50} & 0.40\le x\le 0.9,\\ \dfrac{1.5-x}{0.60} & 0.90\le x\le 1.50,\\ 0 & otherwise,\end{cases}$$

$$\mu_{\tilde{x}_4}(x)=\begin{cases}\dfrac{x-0.90}{0.60} & 0\,.90\le x\le 1.50,\\ \dfrac{2-x}{0.50} & 1.50\le x\le 2,\\ 0 & otherwise,\end{cases}$$

$$\mu_{\tilde{x}_5}(x)=\begin{cases}\dfrac{x-1.5}{0.50} & 1.50\le x\le 2\\ \dfrac{2.50-x}{0.50} & 2\le x\le 2.50\\ 0 & otherwise\end{cases}$$

$$\mu_{\tilde{x}_6}(x)=\begin{cases}\dfrac{x-2}{0.50} & 2\le x\le 2.50\\ \dfrac{4-x}{1.50} & 2.50\le x\le 4\\ 0 & otherwise\end{cases}$$

$$\mu_{\tilde{x}_7}(x)=\begin{cases}\dfrac{x-2.50}{1.50} & 2\,.50\le x\le 4\\ \dfrac{10-x}{6} & 4\le x\le 10\\ 0 & otherwise\end{cases}$$

$$\mu_{\tilde{x}_8}(x)=\begin{cases}\dfrac{x-4}{6} & 4\le x\le 10\\ 1 & x\ge 10\\ 0 & otherwise\end{cases}$$

The Newton-Raphson and the expectation maximization algorithms are employed for computing the maximum likelihood estimates. For the Bayes estimates, non-informative priors with (a,b,c,d)=(0,0,0,0) is considered. The estimates of the three methods are presented in Table 2. From Table 2 shows that the estimates in the three methods are close to each other.

Table 1: times to breakdown of a Capacitor Insulating Fluid dataset

0.35	0.59	0.96	0.99	1.69	1.97	2.07	2.58
2.71	2.90	3.67	3.99	5.35	13.77	25.50	

Table 2: Parameter estimates for the flood dataset sing the proposed methods

Parameter / Method	Estimate of α	Estimate of λ
Newton-Raphson	1.56	1.14
Expectation Maximization	1.50	1.15
Bayes	1.35	0.88

5. Conclusions

The Weibull distribution is a very popular distribution that has been used in different fields such as lifetime analysis, hydrology, engineering, biology, forestry and of course, statistics. Also, its inverse, the inverse Weibull distribution, can be applied in various fields such as biological studies, ecology and medicine branching process. In the literature, several studies presented estimation techniques for the parameters of inverse Weibull distribution based on maximum likelihood, least squares and Bayesian techniques. However, some collected data might be imprecise and therefore each observable event may only be identified with a fuzzy subset of the sample space. Hence, there is a need for suitable statistical methodology to handle these data as well. This paper analyzed times to breakdown dataset using the inverse Weibull distribution when the assumption of the preciseness of the data is not satisfied. Accordingly, this study began by introducing how the classical maximum likelihood estimation and the Bayesian estimation can be utilized to estimate the parameters of the inverse Weibull in presence of imprecise data. In the classical maximum likelihood estimation, both expectation maximization and the Newton-Raphson algorithms have been utilized to find the maximum likelihood estimates. Then all the previous proposed methods have been employed to find estimates of the parameters of the inverse Weibull distribution for the times to breakdown dataset. The study found that the maximum likelihood estimates attained using the Newton Raphson and the expectation maximization are very close to each other. Moreover, the Bayes estimates found based on non-informative priors are close to the corresponding maximum likelihood estimates.

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