A NEW SIMPLE SOLUTION BOUNDARY METHOD FOR POTENTIAL PROBLEMS

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ABSTRACT
The effective evaluation of singular integrals has always been an important issue in boundary element method (BEM), and many effective methods have been developed to handle them. But it is also extremely difficult to solve such problems because of the properties of the method. Based on the existing methods. This work presents a new method for avoiding the calculation of singular integrals, and the paper is the first attempt to apply the new method for the three-dimensional potential problems. There are several benchmark numerical examples indicate the high efficiency and the stability of the proposed technique.

Keywords: *Boundary element method, potential problems, singular integrals, simple solution.*

1. INTRODUCTION

The finite element method has long been a dominant numerical method in many scientific and engineering problems. However, the method requires tedious domain meshing, and for some complex area, the method is often computationally costly. Compared with the FEM, the boundary element method (BEM) [1-4] is nowadays considered as an alternative approach because of its boundary-only discretizations and semi-analytical nature. BEM has been widely used in potential problems, elastic problems, linear problems and nonlinear problems. However, the accuracy of the BEM depend on the evaluation of various singular integrals, and many researchers pay more attention to the effective evaluation of singular integrals [5-10], For example, a general algorithm for calculating singular integrals is proposed in [2], which expresses the non-singular part of integral kernel as a power series of local distance parameters under the intrinsic coordinate system, and entering into analytical calculation. But, in order to determine the coefficients of a power series, we have to solve linear equations, and we have to do this for every unit, and increase the amount of computation. There are many indirect methods have been developed to avoid the calculation of singular integrals [11-19], but this method needs to establish the regular boundary integral equation to avoid the calculation of singular integrals.
Different from the existing work, this paper proposes a new method for calculating strong singular integrals, which is called simple solution boundary element method (SSBEM). It uses a simple solution process to obtain strong singular integral values. There is no need to directly calculate integrals or establish regular boundary integral equations. Therefore, the method has the advantages of simple theory, high calculation efficiency, and accurate results. In addition, the method can calculate any boundary flux $\partial u/\partial x_i\ (i=1,2)$. This method is easy to follow and extend to other problems such as elasticity problems, Stokes problems, Helmholtz problems, etc.

2. BOUNDARY VALUE PROBLEM

In this paper, we always assume that Ω is a bounded domain in R^2, Ω_c its open complement, and Γ their common boundary.

Lemma [18-20]. Let Γ be a piecewise smooth curve (open or closed), and $\hat{x}$ a point on Γ (perhaps a corner). Suppose $h=|\boldsymbol{y}-\hat{\boldsymbol{x}}|$ and $d=\inf_{\boldsymbol{x}\in\Gamma}|\boldsymbol{y}-\boldsymbol{x}|$. If $\psi(\boldsymbol{x})\in C^{0,\alpha}(\Gamma)$ and $\frac{h}{d}\le K_1$ (with constant K_1), then there holds

$$\lim_{\boldsymbol{y}\to\hat{\boldsymbol{x}}}\int_{\Gamma}\frac{x_k-y_k}{|\boldsymbol{x}-\boldsymbol{y}|^2}[\psi(\boldsymbol{x})-\psi(\hat{\boldsymbol{x}})]d\Gamma_{\boldsymbol{x}}=\int_{\Gamma}\frac{x_k-\hat{x}_k}{|\boldsymbol{x}-\hat{\boldsymbol{x}}|^2}[\psi(\boldsymbol{x})-\psi(\hat{\boldsymbol{x}})]d\Gamma_{\boldsymbol{x}}\quad (k=1,2) \tag{1}$$

Consider a two-dimensional potential problem in the domain $\hat{\Omega}\left(\hat{\Omega}=\Omega\ or\ \Omega_c\right)$ governed by the Laplace equation

$$\nabla^2 u(\boldsymbol{x})=0,\quad \boldsymbol{x}=(x_1,x_2)\in\hat{\Omega} \tag{2}$$

with boundary conditions

$$u(\boldsymbol{x})=\bar{u}(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma_1 \tag{3}$$

$$q(\boldsymbol{x})=\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}(\boldsymbol{x})}=\bar{q}(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma_2 \tag{4}$$

where $\Gamma=\Gamma_1\cup\Gamma_2$ is the boundary of $\hat{\Omega}$ with $\Gamma_1\cap\Gamma_2=\varnothing$; $\bar{u}(\boldsymbol{x})$ and $\bar{q}(\boldsymbol{x})$ are the prescribed boundary functions and $\boldsymbol{n}(\boldsymbol{x})$ is the unit outward normal vector at point $\boldsymbol{x}=(x_1,x_2)\in\Gamma$.

3. SIMPLE SOLUTION BOUNDARY ELEMENT METHOD

3.1 regularized indirect boundary integral equations

For potential problem in the domain $\hat{\Omega}\left(\hat{\Omega}=\Omega \text{ or } \Omega_c\right)$ bounded by boundary Γ, in the absence of body source, the equivalent regularized IBIEs for the problem (2)-(4) can be expressed as follows,

$$\int_\Gamma \phi(\boldsymbol{x})d\Gamma=0 \tag{5}$$

$$u(\boldsymbol{y})=\int_\Gamma \phi(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma,\ \ \boldsymbol{y}\in\Gamma \tag{6}$$

For the internal point $\boldsymbol{y}\in\hat{\Omega}$, the integral equations can be written as

$$\frac{\partial u(\boldsymbol{y})}{\partial y_k}=\int_\Gamma[\phi(\boldsymbol{x})-\phi(\hat{\boldsymbol{x}})]\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial y_k}d\Gamma+\phi(\hat{\boldsymbol{x}})\int_\Gamma\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial y_k}d\Gamma,\ \boldsymbol{y}\in\Omega, k=1,2 \tag{7}$$

Let $\boldsymbol{y}\to\hat{\boldsymbol{x}}$, according to the lemma 1, the integral equations can be written as

$$\frac{\partial u(\boldsymbol{y})}{\partial y_k}=\int_\Gamma[\phi(\boldsymbol{x})-\phi(\boldsymbol{y})]\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial y_k}d\Gamma+\phi(\boldsymbol{y})\lim_{y\to\Gamma}\int_\Gamma\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial y_k}d\Gamma,\ \ \boldsymbol{y}\in\Gamma \tag{8}$$

3.2 Simple solution boundary element method (SSBEM) for potential problem

In this section, a new method is proposed. We will take the discrete form of equation (8). Divide the boundary Γ into N segment, and supposed the $\boldsymbol{y}$ at the lth segment, then

$$\frac{\partial u(\boldsymbol{y}_l)}{\partial y_k}=\sum_{\substack{m=1\\m\neq l}}^{N}\int_{\Gamma_m}\phi(\boldsymbol{x})\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y}_l)}{\partial y_k}d\Gamma+\phi(\boldsymbol{y}_l)\lim_{\boldsymbol{y}\to\boldsymbol{x}_l}\int_{\Gamma_l}\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y}_l)}{\partial y_k}d\Gamma,\ \ \boldsymbol{y}_l\in\Gamma_l \tag{9}$$

The integrals in the first term on the right side of the above equation are normal integrals, so the key to the problem lies in how to estimate the second integral on the right side of the above equation.

When constant element interpolation is adopted, the equation (9) can be written as

$$\frac{\partial u(\boldsymbol{y}_l)}{\partial y_k}=\sum_{\substack{m=1\\m\neq l}}^{N}\phi^m\int_{\Gamma_m}\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y}_l)}{\partial y_k}d\Gamma+\phi^l A,\ \ \boldsymbol{y}_l\in\Gamma_l \tag{10}$$

(1) For the finite domain Ω, the simple solution given as

$$u(\boldsymbol{x})=x_1+x_2+1 \tag{11}$$

(2) For the infinite domain Ω_c, the simple solution given as

$$u(\boldsymbol{x})=\frac{x_1}{(x_1-a)^2+(x_2-b)^2} \tag{12}$$

there (a,b) is the point in the outside of Ω_c.

For the problem on $\hat{\Omega}\left(\hat{\Omega}=\Omega \text{ or } \Omega_c\right)$, Substitute the (11) or (12) into the equation (10), and by solving the equations to get the second integral on the right side of the above equation (9).Therefor, the method by solving equations to obtain strong singular integral values. There is no need to calculate integrals or establish regular boundary integral equations.

4. NUMERICAL EXAMPLES

In this section, several test examples are presented to verify the proposed methodology. The feasibility, accuracy and convergence of the SSBEM are carefully investigated. The numerical results obtained using the BEM, which is based on the indirect boundary element formulation with boundary segment being depicted by the real geometry of boundary and the boundary quantities being constant along the elements, are also provided for the purpose of comparison. It is worth mentioning that for all test examples, the real geometry of the domain boundary without approximation is employed for the SSBEM computation. To assess the accuracy of the proposed SSBEM, the relative error of the multiple calculation results is defined by

$$\text{Relative error (RE)} = \sqrt{\sum_{k=1}^{M}(u_{num}^{k} - u_{exact}^{k})^2} \Big/ \sqrt{\sum_{k=1}^{M}(u_{exact}^{k})^2}$$

where M is the total number of calculation points, u_{num}^{k} and u_{exact}^{k} exact denote the numerical and exact solution at the kth calculation point, respectively.

Test problem 1: A square domain with Dirichlet discontinuous boundary conditions

First, as shown in fig 1, we investigate a square domain $[0,1]\times[0,1]$ subject to the Dirichlet discontinuous boundary conditions given as follows:

$$u(x_1,0) = x_1, u(x_1,1) = u(0,x_2) = u(1,x_2) = 0$$

An analytical solution is available as follows:

$$u(x_1,x_2) = \sum_{n=1}^{\infty} D_n \sinh\left(n\pi(1-x_2)\right)\sin\left(n\pi x_1\right)$$

where

$$D_n = \frac{2(-1)^{n+1}}{(n\pi)\sinh(n\pi)}$$

There are 80 boundary elements to calculate the example, Fig. 2(a) and (b) show the contour plots of the field potential solutions by using the analytical results and the SSBEM, respectively. It can be seen that results obtained by SSBEM match the exact solutions very well. As shown in fig 3, there are 400 points in $0.15 \le x_1, x_2 \le 0.85$, and given the error surface at different boundary elements.

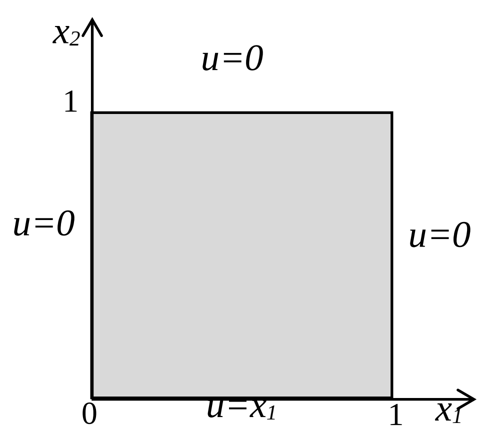
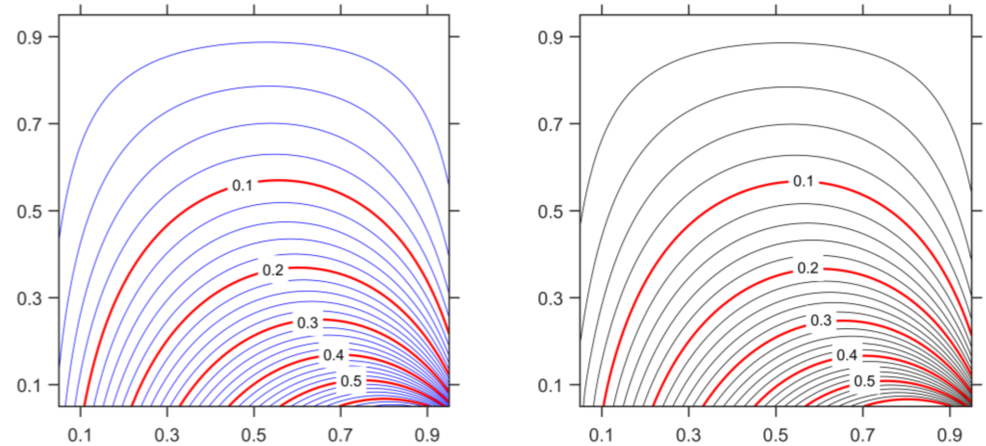
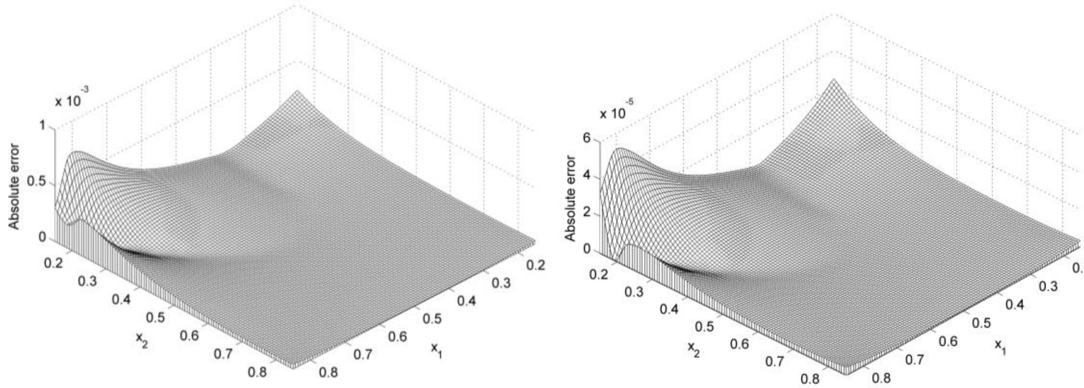


Fig 1.the square domain



(a) *(b)*

Fig 2. The field potential solutions: (a) exact results and (b) proposed method



(a) *(b)*

Fig 3. Absolute error surfaces for field potential solutions, where the number of boundary element is 40 for (a) and 80 for (b), respectively.

Test problem 2: Mixed boundary conditions for irregular regions

In this case, an irregular domain subject to the mixed type boundary conditions is considered as fig 4, and the boundary conditions are complex and oscillatory. Problem sketch are shown in Fig.4, respectively, in which the parametric representation of the boundary $\Gamma = \Gamma_1 \bigcup \Gamma_2$ where

$$\Gamma_1 = \left\{ (R\cos\varphi, R\sin\varphi) : R = e^{\sin\varphi}(\sin^2(2\varphi)) + e^{\cos\varphi}(\cos^2(2\varphi)), 0 \le \varphi \le 2\pi \right\}$$

$$\Gamma_2 = \left\{ (r\cos\theta + 0.5, r\sin\theta + 0.5) : r = 0.4, \pi \le \theta \le 2\pi \right\}$$

The boundary conditions are

$$\bar{u} = e^{x_1}\cos x_2 \qquad on\ \ \Gamma_1$$

$$\bar{q} = e^{x_1} n_1 \cos x_2 - e^{x_1} n_2 \sin x_2 \quad on\ \ \Gamma_2$$

The analytical solution of this problem is given by

$$u(x_1, x_2) = e^{x_1}\cos x_2$$

In the numerical calculation, the boundary is divided into 80 units and 40 units. Figure 5 is the isopotential diagram of the exact solution and the numerical solution of the inner point temperature. It can be seen that the numerical results are very consistent with the exact solution. Fig 6 shows the relative error curve of the heat flow at the inner boundary node. It can be seen that the simple solution method accurately calculates the temperature gradient at the boundary node and effectively avoids the influence of the boundary singular integral.

In addition, as shown in figure 8, with the increase of the number of boundary elements, the calculation error of boundary node temperature becomes smaller and smaller, indicating that the method in this paper has good stability for solving the oscillation problem in complex regions.

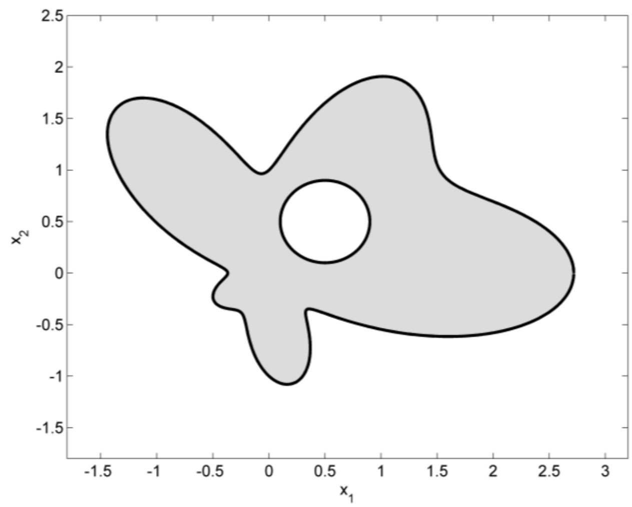
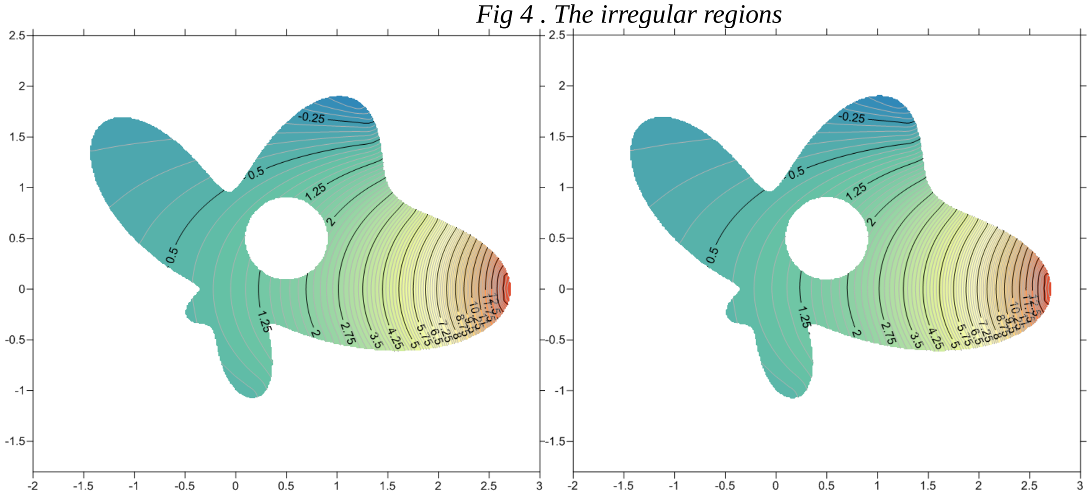


Fig 4 . The irregular regions



(a) *(b)*

Fig 5. The field potential solutions: (a) exact results and (b) proposed method

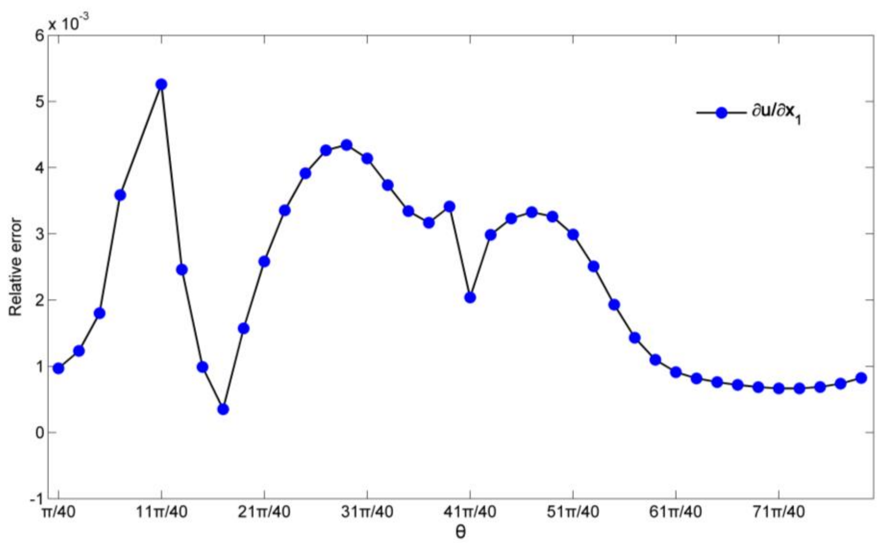


Fig 6. The relative errors for the flux $\partial u/\partial x_1$ *on the boundary* Γ_2

Test problem 3: 2D multiply-connected domain with mixed boundary conditions

In this case, a multiply-connected domain subject to the mixed type boundary conditions is considered. The problem sketch and the boundary conditions are shown in Fig. 7.

$$\bar{u}(x_1, x_2) = x_1^2 - x_2^2 + x_2, \quad \bar{q}(x_1, x_2) = 2x_1 n_1 + (-2x_2 + 1)n_2$$

The Γ_1 is divided into 36 elements and Γ_2 is divided into 24 elements. Table 1 gives the internal points numerical solutions and the analytical solutions, and fig 8 gives the relative error of temperature u and flux q on the boundary. As shown that the method in this paper has good stability for solving the oscillation problem in complex regions.

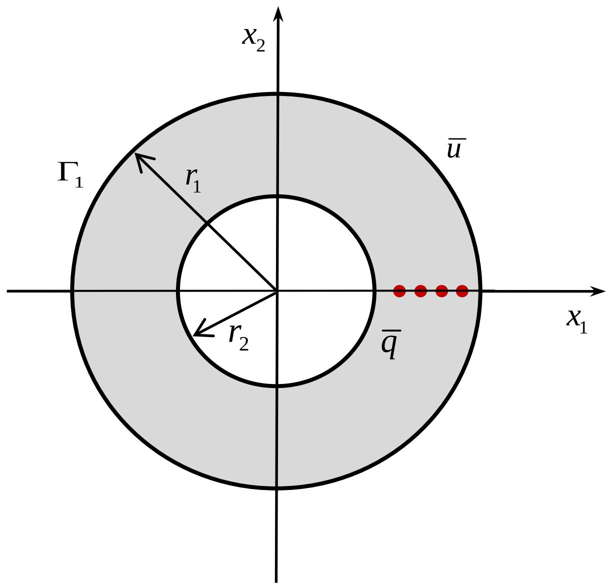


Fig 7. Problem sketch

Table 1 Temperature solutions on internal points

Internal Points	Exact solutions	Numerical results	Relative error
(6.0, 0.0)	0.3600000E+02	0.3598754E+02	3.462057E-03
(7.0, 0.0)	0.4900000E+02	0.4897589E+02	4.920773E-03
(8.0, 0.0)	0.6400000E+02	0.6396386E+02	5.646912E-03
(9.0, 0.0)	0.8100000E+02	0.8094677E+02	6.571639E-03

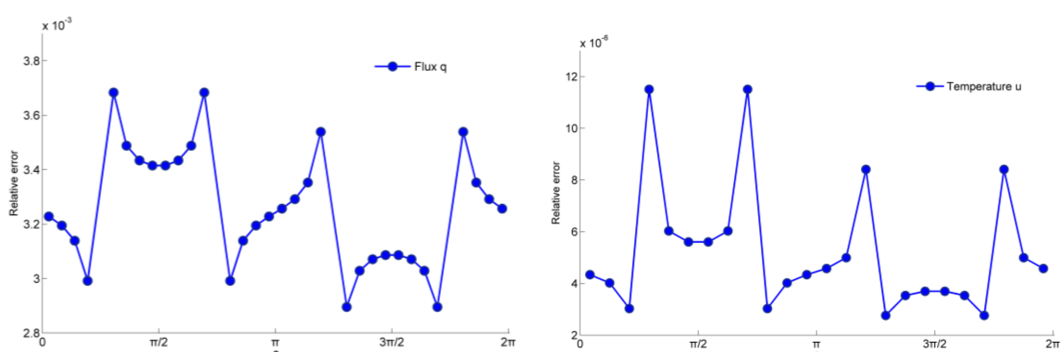


Fig 7. The relative error of boundary temperature u and flux q

5. CONCLUSION

In this paper, a new simple solution boundary element method (SSBEM) is proposed to solve 2-D potential problems. This method is based on the regularized indirect boundary integral equations (RIBIE). However, the new method by solving the equations to avoid calculating strong singular integral, and it has a simple mathematical theory, easy programming, high accuracy, etc. Three numerical examples verify the effectiveness of the method. Moreover, it is observed that for the boundary value problem with discontinue boundary conditions, the SSBEM is better than traditional BEM in the theory and program.

5. REFERENCES

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