STOCHASTIC OBJECTIVE FUNCTION OF LP PROBLEM WITH GE (α, λ, μ) DISTRIBUTED RANDOM PARAMETERS

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ABSTRACT

In this paper, the probabilistic linear programming model with random objective function and deterministic structural constraints- is transformed into an equivalent deterministic model assuming that some or all coefficients in objective function are independent random parameters and follow the generalized exponential distribution $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$. Firstly, the moment generating function (MGF), expectation, and variance of $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$ are derived through theorem (1). This theorem gives the general case of what presented by Gupta and Kundu. Secondly, an equivalent deterministic model is introduced using the expected value (E- model) criterion and minimum variance (v-model) criterion. Finally, a numerical example is introduced to illustrate the transformation from probabilistic into equivalent deterministic model.

Keywords: *Moment Generating Function, Expectation, Variance, $GE(\alpha_j, \lambda_j, \mu_j)$, Probabilistic Programming.*

1. INTRODUCTION

Many studies presented different criteria for converting a probabilistic objective function into an equivalent deterministic one assuming different probability distributions, the most frequently used criteria are those introduced by Charnes and Cooper [1] who introduced three criteria: the expected value criterion (E- model), minimum variance criterion (V-model), and maximum probability criterion (P-model) for converting a probabilistic objective function into an equivalent deterministic one. Many researchers manipulated probabilistic objective functions based on these criteria assuming different distributions. For instance, Biswal et al [10] used the expected value criterion (E-model) in order to obtain the equivalent deterministic objective function assuming that the coefficients of decision variables in the objective function are independent random variables that follow a single parameter exponential distribution. Acharaya and Biswal [17] used the E-model assuming that the coefficients of decision variables in the objective function are independent random variables that follow the normal distribution. Also, Turgut and Murat [13] used the expected value criterion and the variance criterion assuming that the parameters- in a multi-objective model, take discrete values with different probabilities. Ismail et al [8] presented an equivalent deterministic model to the probabilistic programming model when the parameters of the objective function are independent random variables distributed as the two-parameter Weibull distribution using the E-model criterion.

Gupta and Kundu [16] introduced the Generalized Exponential Distribution $\mathrm{GE}(\alpha, \lambda, \mu)$ which is considered the general case of both one-parameter and two-parameter exponential distributions. the Generalized Exponential Distribution has many advantages and preferable characteristics compared to other non-negative distributions, it is considered more flexible than the gamma and chi-square distributions because its shape parameter can take integer or non-integer values. Besides; both cumulative function and inverse cumulative function of the $\mathrm{GE}(\alpha, \lambda, \mu)$ are in closed-form [14]. Also, the $\mathrm{GE}(\alpha, \lambda, \mu)$ has shown efficiency in many applications such as in forecasting precipitation records [9]. In estimating the average life of power system equipment [6], As well as checking the performance of healing models [12].

Many studies used the $\mathrm{GE}(\alpha, \lambda, \mu)$ in the framework of probabilistic programming. El-Dash [2] introduced the inverse cumulative function of the $\mathrm{GE}(\alpha, \lambda, \mu)$, and provided the equivalent deterministic model of the probabilistic model using the chance constrained programming technique when some of the right-hand side (RHS) parameters, or when one of the left-hand side (LHS) parameters of the constraints are distributed as $\mathrm{GE}(\alpha, \lambda, \mu)$, and the model is solved using the simplex method. El-Dash and Hafez [4] have proposed an iterative approach to obtain the equivalent deterministic model of the probabilistic model when the RHS parameters of the constraints are independent random variables that follow the $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$ in the case of joint constraints. El-Dash [3] presented a proposed method for transforming the probabilistic linear goal programming model to an equivalent deterministic model when some or all of the parameters of the RHS of the constraints are random variables that follow the $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$. This method provides the best compromise solution to the problem. El-Khabeary et al. [7] have proposed an equivalent deterministic model of the probabilistic model when two of LHS parameters of constraints are independent random variables that follow the bivariate $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$.

In this paper we are interested in transforming a probabilistic programming model into an equivalent deterministic one when some or all of the parameters of the objective function are independent $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$ random parameters. The E-model and V-model will be used in this transformation. Gupta and Kundu [16] derived the moment generating function, expectation, and variance of $\mathrm{GE}(\alpha_j)$ distribution, then in 2001 they derived the previous functions for $\mathrm{GE}(\alpha_j, \lambda_j)$ distribution.

Therefore; In this paper, firstly; the moment generating function, expectation, and variance of $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$ are derived. Secondly; the derived functions are used in constructing both the expected value (E- model) criterion and the minimum variance (v-model) criterion, in order to convert a probabilistic objective function with some random parameters following the $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$ to an equivalent deterministic function.

2. THE MOMENT GENERATING FUNCTION OF GE($\alpha_j, \lambda_j, \mu_j$).

In this section, we present the equivalent deterministic model for a probabilistic linear model assuming that some or all coefficients $\tilde{c_j}$ in the objective function are random and independently distributed as $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$, $j = 1, \dots, n$, and the structural constraints are linear and deterministic. Now; let:

$$\text{Min. } \tilde{z} = \sum_{j=1}^{n} \tilde{c_j} x_j \tag{2.1}$$

$$S.T \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad ; i = 1, 2, \dots, m \tag{2.2}$$

$$x_j \ge 0 \quad ; j = 1, 2, \dots, n \tag{2.3}$$

Where; x_j; $j = 1, 2, \dots, n$. are the decision variables, $\tilde{c_j}$; $j = 1, \dots, n$ are independent random parameters following $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$, a_{ij} is the j^{th} deterministic L.H.S coefficient of the i^{th} constraint, and b_i is the deterministic R.H.S parameter of the i^{th} constraint .

For model (2.1)-(2.3), we assume that the $\tilde{c_j}'s$ are independent random parameters and follow the $\mathrm{GE}((\alpha_j, \lambda_j, \mu_j), j = 1, \dots, n$. In order to obtain the equivalent deterministic objective function of (2.1) based on the E-model or the V-model, it is required to find at first the expectation, and variance as introduced in theorem (1), where the PDF of $\tilde{c_j}$ is given by:

$$f(\tilde{c_j}) = \alpha_j \lambda_j (1 - e^{-\lambda_j(\tilde{c_j} - \mu_j)})^{\alpha_{ij} - 1} e^{-\lambda_j(\tilde{c_j} - \mu_j)}; \ \tilde{c_j} > \mu_j \ ; \alpha_j, \lambda_j, \mu_j > 0 \ , j = 1, \dots, n \tag{2.4}$$

<u>Theorem (1)</u>. Consider the random variables $\tilde{c_j}$, $j = 1, \dots, n$ in (2.1) with density function $f(\tilde{c_j})$ defined in (2.4), then:

1- For $\tilde{c_j}$, $j = 1, \dots, n$

i. The MGF of $\tilde{c_j}$ of the $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$ is

$$M_{\tilde{c_j}}(t) = \alpha_j e^{t\mu_j} \mathrm{B}(\alpha_j, 1 - \frac{t}{\lambda_j}) = \frac{e^{t\mu_j} \alpha_j \Gamma(\alpha_j) \Gamma(1 - \frac{t}{\lambda_j})}{\Gamma(\alpha_j - \frac{t}{\lambda_j} + 1)} \tag{2.5}$$

ii. expectation of $\tilde{c_j}$ of the $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$ is

$$\mathrm{E}(\tilde{c_j}) = \mu_j + \frac{1}{\lambda_j}\left[\Psi(\alpha_j + 1) - \Psi(1)\right] \tag{2.6}$$

iii. expectation of $\tilde{c_j}$ of the $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$ is

$$\mathrm{V}(\tilde{c_j}) = \frac{1}{\lambda_j^2}\left[\Psi'(1) - \Psi'(\alpha_j + 1)\right] \tag{2.7}$$

Where: B (.,.) Represents the Beta function, Γ (.) Represents the Gamma function, $\Psi(\alpha_j)$ represents the digamma function, and $\Psi'(\alpha_j)$ represents the derivative of the digamma function [5].

2- the equivalent deterministic objective function of (2.1), according to E- model criterion and V-model criterion, respectively, are as follows:

$$\text{Min. } \tilde{z} = \mathrm{E}(\tilde{z}) = \sum_{j=1}^{n} x_j \left[\mu_j + \frac{1}{\lambda_j}\left[\Psi(\alpha_j + 1) - \Psi(1)\right]\right] \tag{2.8}$$

$$\text{Min. } \tilde{z} = \mathrm{V}(\tilde{z}) = \sum_{j=1}^{n} x_j^2 \left[\frac{1}{\lambda^2}\left[\Psi'(1) - \Psi'(\alpha + 1)\right]\right] \tag{2.9}$$

Proof: firstly, since the MGF of the random variable $\tilde{c_j}$ takes the following form:

$$M_{\tilde{c_j}}(t) = E\left(e^{t\tilde{c_j}}\right) = \int_{\mu_j}^{\infty} e^{t\tilde{c_j}}\, \alpha_j\lambda_j\left(1 - e^{-\lambda_j(\tilde{c_j}-\mu_j)}\right)^{\alpha_j-1} e^{-\lambda_j(\tilde{c_j}-\mu_j)}\, d\tilde{c_j} \qquad (2.10)$$

by using the substitution technique, let:

$$y = e^{-\lambda_j(\tilde{c_j}-\mu_j)} \qquad ; \therefore\ \tilde{c_j} = \mu_j - \frac{\ln(y)}{\lambda_j} \qquad (2.11)$$

$$\frac{dy}{d\tilde{c_j}} = -\lambda_j e^{-\lambda_j(\tilde{c_j}-\mu_j)} = -\lambda_j y \qquad ; \therefore\ d\tilde{c_j} = \frac{dy}{-\lambda_j y} \qquad (2.12)$$

From (2.11)- (2.12), If $\tilde{c_j} \to \infty$ then $y \to 0$, and if $\tilde{c_j} \to \mu_j$ then $y \to 1$ .Thus, the mgf in (2.10) can be rewritten as follows:

$$M_{\tilde{c_j}}(t) = e^{t\mu_j}\alpha_j \int_0^1 e^{-t\frac{\ln(y)}{\lambda_j}} (1-y)^{\alpha_j-1}\, dy = e^{t\mu_j}\alpha_j \int_0^1 y^{\frac{-t}{\lambda_j}} (1-y)^{\alpha_j-1}\, dy \qquad (2.13)$$

The previous integral represents the beta function, and it can be placed in the form of a gamma function (because it is easier to use it later to find expectation and variance of $\tilde{c_j}$) as follows:

$$M_{\tilde{c_j}}(t) = \alpha_j e^{t\mu_j}\, B\left(\alpha_j, 1 - \frac{t}{\lambda_j}\right) = \frac{e^{t\mu_j}\alpha_j \Gamma(\alpha_j)\Gamma(1-\frac{t}{\lambda_j})}{\Gamma(\alpha_j - \frac{t}{\lambda_j} + 1)} \qquad ▪$$

which is the MGF of $GE(\alpha_j, \lambda_j, \mu_j)$ as indicated in equation (2.5).

Secondly, to find the expectation and variance of the $GE(\alpha_j, \lambda_j, \mu_j)$ distribution using the mgf in (2.5), the digamma function $\Psi(\alpha_j)$ as well as its derivative $\Psi'(\alpha_j)$ will be used. Those functions take the following form [5].

$$\Psi(\alpha_j) = \frac{d}{d\alpha_j} \ln\left(\Gamma(\alpha_j)\right) = -a - \frac{1}{\alpha_j} + \sum_{k=1}^{\alpha_j} \frac{1}{k} \qquad (2.14)$$

$$\Psi'(\alpha_j) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha_j)^2} \quad ; \alpha_j > 1 \qquad (2.15)$$

Where: a Represents the Euler's Constant and takes the following approximate value:

$$a = -\int_0^{\infty} e^{-\tilde{c_j}} \ln \tilde{c_j}\, d\tilde{c_j} = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right)$$
$$= 0.57721566490153286060\ldots \cong 0.5772 \qquad (2.16)$$

1) Expectation $E(\tilde{c_j})$

By taking the logarithm of the natural basis of the MGF (Ln MGF) in (2.5), we will obtain the following form:

$$\varphi(t) = LnM_{\tilde{c_j}}(t) = Ln\, \frac{e^{t\mu_j}\, \alpha_j\Gamma(\alpha_j)\Gamma(1-\frac{t}{\lambda_j})}{\Gamma(\alpha_j - \frac{t}{\lambda_j} + 1)}$$

$$= Lne^{t\mu_j} + Ln\alpha_j + Ln\,\Gamma(\alpha_j + 1) + Ln\,\Gamma\left(1 - \frac{t}{\lambda_j}\right) - Ln\,\Gamma(\alpha_j - \frac{t}{\lambda_j} + 1) \qquad (2.17)$$

By taking the first derivative of (2.17):

$$\frac{d}{dt}\varphi(t) = \frac{d}{dt}\left\{Ln\, e^{t\mu_j} + Ln\alpha_j + Ln\,\Gamma(\alpha_j+1) + Ln\,\Gamma\left(1 - \frac{t}{\lambda_j}\right) - Ln\,\Gamma(\alpha_j - \frac{t}{\lambda_j} + 1)\right\} \qquad (2.18)$$

$$\frac{d}{dt} Ln\, e^{t\mu_j} = \mu_j; \qquad \frac{d}{dt}\left(Ln\alpha_j + Ln\,\Gamma(\alpha_j+1)\right) = 0 \qquad (2.19)$$

$$\frac{d}{dt} Ln\,\Gamma\left(1 - \frac{t}{\lambda_j}\right) = \frac{\Gamma'(1-\frac{t}{\lambda_j})}{\Gamma\left(1-\frac{t}{\lambda_j}\right)}\left(\frac{-1}{\lambda_j}\right) = \Psi\left(1 - \frac{t}{\lambda_j}\right)\left(\frac{-1}{\lambda_j}\right) \qquad (2.20)$$

Where: $\Gamma'(.)$ Represents the derivative of gamma function, and

$$\frac{d}{dt} Ln\,\Gamma(\alpha_j - \frac{t}{\lambda_j} + 1) = \frac{\Gamma'(\alpha_j - \frac{t}{\lambda_j} + 1)}{\Gamma(\alpha_j - \frac{t}{\lambda_j} + 1)}\left(\frac{-1}{\lambda_j}\right) = \Psi\left(\alpha_j - \frac{t}{\lambda_j} + 1\right)\left(\frac{-1}{\lambda_j}\right) \qquad (2.21)$$

Thus, by substituting (2.19) - (2.21) in (2.18), the first derivative in (2.18) becomes as follows:

$$\frac{d}{dt}\varphi(t) = \mu_j + \Psi\left(1 - \frac{t}{\lambda_j}\right)\left(\frac{-1}{\lambda_j}\right) - \Psi\left(\alpha_j - \frac{t}{\lambda_j} + 1\right)\left(\frac{-1}{\lambda_j}\right) \qquad (2.22)$$

$$\frac{d}{dt}\varphi(t) = \mu_j + \frac{1}{\lambda_j}\Psi\left(\alpha_j - \frac{t}{\lambda_j} + 1\right) - \frac{1}{\lambda_j}\Psi\left(1 - \frac{t}{\lambda_j}\right) \qquad (2.23)$$

After substituting t = 0 into (2.23), we obtain the expectation as follows:

$$\mathrm{E}(\tilde{c_j}) = \mu_j + \left(\frac{1}{\lambda_j}\right)\left[\Psi(\alpha_j + 1) - \Psi(1)\right] \quad ■$$

2) Variance $V(\tilde{c_j})$

To find the variance, we will take the first derivative of (2.23) as follows:

$$\frac{d^2}{dt^2}\varphi(t) = \frac{d}{dt}\left\{\mu_j + \frac{1}{\lambda_j}\Psi\left(\alpha_j - \frac{t}{\lambda_j} + 1\right) - \frac{1}{\lambda_j}\Psi\left(1 - \frac{t}{\lambda_j}\right)\right\} \tag{2.24}$$

Where;

$$\frac{d}{dt}\mu_j = 0 \;;\; \frac{d}{dt}\frac{1}{\lambda_j}\Psi\left(\alpha_j - \frac{t}{\lambda_j} + 1\right) = \frac{1}{\lambda_j}\Psi'\left(\alpha_j - \frac{t}{\lambda_j} + 1\right)\left(\frac{-1}{\lambda_j}\right) \tag{2.25}$$

$$\frac{d}{dt}\frac{1}{\lambda}\Psi\left(1 - \frac{t}{\lambda_j}\right) = \frac{1}{\lambda_j}\Psi'\left(1 - \frac{t}{\lambda_j}\right)\left(\frac{-1}{\lambda_j}\right) \tag{2.26}$$

Thus, by substituting (2.25)- (2.26) in (2.24),

$$\frac{d^2}{dt^2}\varphi(t) = \frac{1}{\lambda_j}\Psi'\left(\alpha_j - \frac{t}{\lambda_j} + 1\right)\left(\frac{-1}{\lambda_j}\right) - \frac{1}{\lambda_j}\Psi'\left(1 - \frac{t}{\lambda_j}\right)\left(\frac{-1}{\lambda_j}\right) \tag{2.27}$$

$$\frac{d^2}{dt^2}\varphi(t) = \frac{-1}{\lambda_j^{2}}\Psi'\left(\alpha_j - \frac{t}{\lambda_j} + 1\right) + \frac{1}{\lambda_j^{2}}\Psi'\left(1 - \frac{t}{\lambda_j}\right) \tag{2.28}$$

After substituting t = 0 into (2.28), the variance becomes as follows:

$$V(\tilde{c_j}) = \frac{1}{\lambda_j^{2}}\left[\Psi'(1) - \Psi'(\alpha_j + 1)\right] \quad ■$$

Finally, according to the E- model criterion and V-model criterion the equivalent deterministic objective function of probabilistic objective function in (2.1) are as follows, respectively:

$$\text{Min. } \tilde{z} = E(\tilde{z}) = \sum_{j=1}^{n} x_j\, E(\tilde{c_j}) \tag{2.29}$$

$$\text{Min. } \tilde{z} = V(\tilde{z}) = \sum_{j=1}^{n} x_j^2\, V(\tilde{c_j}) \tag{2.30}$$

Hence, by substituting (2.6)- (2.7) in (2.29)- (2.30), we obtain the equivalent deterministic objective function of (2.1) as given in (2.8)- (2.9).

3. SPECIAL CASES

1) let $\lambda_j = 1, \mu_j = 0$ in (2.5)- (2-7), then:

1) $$M_{\tilde{c_j}}(t) = \alpha_j\, B(\alpha_j, 1 - t) = \frac{\alpha_j \Gamma(\alpha_j)\Gamma(1-t)}{\Gamma(\alpha_j - t + 1)} \tag{3.1}$$

2) $$E(\tilde{c_j}) = \Psi(\alpha_j + 1) - \Psi(1) \tag{3.2}$$

3) $$V(\tilde{c_j}) = \Psi'(1) - \Psi'(\alpha_j + 1) \tag{3.3}$$

These results are the same as the results provided by Gupta and Kundu [16] for generalized exponential distribution with one parameter $GE(\alpha_j)$.

(2) let $\mu_j = 0$ in (2.5)- (2-7), then:

1) $$M_{\tilde{c_j}}(t) = \alpha_j\, B\left(\alpha_j, 1 - \frac{t}{\lambda_j}\right) = \frac{\alpha_j \Gamma(\alpha_j)\Gamma\left(1 - \frac{t}{\lambda_j}\right)}{\Gamma\left(\alpha_j - \frac{t}{\lambda_j} + 1\right)} \tag{3.4}$$

2) $$E(\tilde{c_j}) = \frac{1}{\lambda_j}\left[\Psi(\alpha_j + 1) - \Psi(1)\right] \tag{3.5}$$

3) $$V(\tilde{c_j}) = \frac{1}{\lambda_j^{2}}\left[\Psi'(1) - \Psi'(\alpha_j + 1)\right] \tag{3.6}$$

These results are the same as the result provided by Gupta and Kundu [15] for generalized exponential distribution with two parameters $GE(\alpha_j, \lambda_j)$.

4. NUMERICAL EXAMPLE

In this section, we introduced a numerical example to illustrate the procedure of the transformation from probabilistic linear programming model into an equivalent deterministic model, when parameters in the objective function follow the $GE(\alpha_j, \lambda_j, \mu_j)$, and structural constraints are deterministic. Consider the following probabilistic model:

$$\text{Min. } \tilde{Z} = \tilde{c_1}x_1 + \tilde{c_2}x_2 \tag{4.1}$$

S.t

$$x_1 - x_2 \le 2 \tag{4.2}$$

$$2x_1 + x_2 \ge 10 \tag{4.3}$$

$$3x_1 + 4x_2 \le 24 \tag{4.4}$$

$$x_1, x_2 \ge 0 \tag{4.5}$$

Where; x_1, x_2 are decision variables, $\tilde{c_1}, \tilde{c_2}$ are independent random parameters following GE distribution with the following parameters

$\tilde{c_1} \sim \mathrm{GE}(\alpha_1 = 2, \lambda_1 = 1, \mu_1 = 2)$ and $\tilde{c_2} \sim \mathrm{GE}(\alpha_2 = 2, \lambda_2 = 2, \mu_2 = 3)$ (4.6)

based on Theorem (1) and by substituting values of the parameters (4.6) of random variable $\tilde{c_1}, \tilde{c_2}$ in (2.8)- (2-9), then the equivalent deterministic objective function of (4.1) is as follows, according to:

1) the E- model criterion:

Min. $\tilde{z} = E(\tilde{z}) = 3.5\,x_1 + 3.75x_2$ (4.7)

Hence, the linear equivalent deterministic programming model in this case becomes (4.7) and (4.2) - (4.5), which can be solved by using the Simplex method, and the optimal solution is as follows:

$E(\tilde{Z}) = 21.5$, $x_1 = 4, x_2 = 2$ (4.8)

2) the V-model criterion:

Min. $\tilde{z} = V(\tilde{z}) = 1.25\,x_1^{\,2} + 0.3125\,x_2^{\,2}$ (4.9)

Hence, the equivalent deterministic programming model in this case becomes (4.9) and (4.2) - (4.5), which is a nonlinear programming model with convex objective function that can be solved by using the quadratic programming, and the optimal solution is as follows:

$V(\tilde{Z}) = 16.85$, $x_1 = 3.2, x_2 = 3.6$ (4.10)

Thus, the summary of solutions for the probabilistic model (4.1) - (4.5) according to the two criteria is as follows:

Table 1. the summary of solutions for the probabilistic model

criterion	Objective function	decision variables
E- model	$E(\tilde{Z}) = 21.5$	$x_1 = 4, x_2 = 2$
V- model	$V(\tilde{Z}) = 16.85$	$x_1 = 3.2, x_2 = 3.2$

It is apparent from the above table that the best solution for the probabilistic programming model results from V-model criterion.

5. CONCLUSION

In this paper, we obtained the moment generating function, expectation, and variance of $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$ through theorem (1). Then; by using it the expected value criterion (E- model) and the variance criterion (v-model), the equivalent deterministic model of the probabilistic programming model under the assumption that some or all random parameters of the objective function follow $\mathrm{GE}(\alpha_j, \lambda_j, \mu_j)$ is presented. Also, some special cases of the properties of the distribution are introduced, where these cases satisfy the results provided by Gupta and Kundu (1999, 2001).

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