ON THE SENSITIVITY STUDY OF MODIFIED BREUSCH PAGAN IN THE CLASS OF EXISTING HETEROSCEDASTICITY TESTS IN NONLINEAR MODELS

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ABSTRACT

It has been established that in the presence of heteroscedasticity, the least squares method has two major drawbacks: inefficient parameter estimate and biased variance. Tests for heteroscedasticity had been developed for linear models but not for nonlinear models. This study is aimed at investigating the Power of some tests for detecting heteroscedasticity in nonlinear models namely; Cobb-Douglas (CD) and Constant Elasticity of Substitution (CES). The objectives of the study were to propose a test by modifying existing Breusch-Pagan test for detecting heteroscedasticity; and compare the performance, in term of the test, the Modified Breusch-Pagan and some existing tests of heteroscedasticity for the models.

The CD were transformed to intrinsically linear models through logarithms while the linearization by Kmenta was used for CES production function model. Using the parameter estimates of the model, the residuals were computed and used as the dependent variable for the auxiliary regression. The error structured data were drawn from a normal distribution with mean, zero and variance, σ^2 . Real life data from World Development Indicator was used for gross domestic product (GDP), Capital (K) and Labour (L) to confirm the results from simulation study.

The findings of the study showed that for the level of heteroscedasticity in the models considered, the proposed Modified Breusch-Pagan (MBP) tests are the most powerful for all the sample sizes because it offered substantial improvements over Breusch-Pagan, White and Parks tests.
It is therefore recommended that the proposed Modified Breusch-Pagan and Park tests should be used for nonlinear models.

Keywords: *CD, CES, Heteroscedasticity, Linearization, Intrinsically, Power of the test.*

1. Introduction

Non-linear models are utilized for analyzing demand and production functions. A simple Cobb-Douglass production function cannot be transformed into linearity if the error term is added rather than multiplied [1]. Econometric modeling requires the incorporation of disturbance term (an error term) as well as the specification of its distribution. The specification of error terms is the main problem in applied econometrics. The functional form y in the model cannot be decided in isolation from the specification of error term. The nonlinear regression model technique is appropriate when a linear regression model cannot adequately represent the relationship between variables. There is heteroscedasticity, or unequal, or nonconstant variance if the variance of u_i is σ_i^2, which implies that it varies from observation to observation. An OLS estimator remains unbiased however becomes inefficient if the errors are unequal [2,3]. It is widely known that the estimates of the standard errors are not consistent, and can either be too large or small resulting in incorrect inferences. The consequence of the violation of homoscedasticity which is one of the assumptions of CLRMs, leads to inefficient in estimating regression parameters. Hence, the need arise to examine the existence of unequal variance in data sets. The need to have relatively larger sample sizes when testing for heteroscedasticity so as to determine the nature of such errors and appraise their behaviours necessitated this study. This is often impracticable in real life situations with cross sectional data. Hence, various studies have been undertaken using the Monte Carlo simulation; in an attempt to shed more light on this problem. Thus, this study examined the performance of many tests for heteroscedasticity in a nonlinear model for small and large samples.

A linear function of the unknown parameters in an econometric model is called a regression function. Consequently, in a nonlinear model, the regression function could be given as nonlinear function of the unknown parameters.

Nonlinear models are classified into two groups: the presence or non-presence of a linear relation with respect to the parameters. The models can be intrinsically linear or nonlinear. A model that is parameter linear and also linear or not in variables is called linear model. Likewise, a parameter nonlinear and linear or not in variables is called nonlinear model. Some models might appear nonlinear within the parameters but are intrinsically linear as a result of appropriate modification such models could be linear in the parameter. However, assuming the parameters of these models could not be transformed linear; they are known as intrinsically nonlinear regression models.

Taylor series expansion is the main approach utilized in linearising nonlinear equation. When the Taylor series expansion model is applied to estimate nonlinear regression, it uses search procedures called Gauss-Newton and Newton-Raphson methods. Other direct search procedures besides these are the methods of Steepest Descent and Marquardt Algorithm.

White (1980) discovered the asymptotically form of the Heteroscedasticity Consistent Covariance Matrix (HCCM), called hereafter as HCO. In subsequent paper, Mackinnon and White (1995) worried on the result of HCO in low samples and discovered three different estimators referred to as HC1, HC2 and HC3. But the three estimators are asymptotically equal to HC0. Meanwhile, better properties in finite samples were anticipated. [4] presented that the data utilized by [5] had high leverage with one observation. Once this observation was eliminated and simulation was performed several times, they discovered that every version of HCCM performed well. This result is identified with the research of [6,7,8,9,10].

The issue of investigating the homoscedasticity in linear model was considered by [11]. This is based on the notion that there is a noticeable drawback once the model contains many irrelevant parameters. They proposed two enhancements to the standard likelihood ratio heteroscedasticity test. [12] obtained a Bartlett correction test statistic that is distributed as χ^2 to the accuracy of second degree and [13] applied the modified profile likelihood projected by [14] of equal variances to the LRT.

[15] described an answer to regulating issue on the size of homoscedasticity test in linear regression problem. They studied methods established on the standard test statistic such as Goldfeld-Quandt, Cochran, Hartley, Glejser, Barlett, Breuch-Pagan, Szroeter and White as well as test auto-regressive conditional heteroscedasticity. They also suggested several extensions of the existing procedure. [16] recognized the fact that [17] test is one of the most popular tests that can be effectively applied to large sample sizes. Conversely, the asymptotic χ^2 distribution in White’s statistic is not as effective in providing an approximation of finite sample distribution. The size and strength of White’s test in low samples may not be satisfactory. They proposed a resampling method to enhance White’s test small properties called bootstrap procedure.

Two tests for homoscedasticity that need very little information of the functional relationship establishing the variance of the error term was proposed by [18]. Approximating the true relationship by Tailor’s series expansion was the concept of the first test that is essentially linearizing the function in a neighbourhood. [19] earlier applied this idea to non-linear variable selection, while [20] focused on causality testing in a non-linear framework. [21] also compared the power in small samples of several tests for conditional heteroscedasticity in which two new tests established on neural networks are projected: the main interest in them arises from the actual fact that they do not require the precise specification of the conditional variance beneath the choice.

[22] presented a short-term review of nonlinear models of autoregressive conditional heteroscedasticity. The models in question are parametric nonlinear extensions of the first model by [23]. After presenting the different models, linearity testing and parameter estimation are argued, forecasting volatility with nonlinear models is reflected. Finally, parametric nonlinear models established on multiplicative decomposition of the variance received consideration. [24] analysed heteroscedasticity checks for models of single index, in their paper two test statistics are suggested through quadratic conditional moments. Without the usage of dimension reduction structure, in the nonparametric sense, the first test has the normal convergence rate. With the use of dimension reduction structure of mean and variance functions, the subsequent one has quicker convergence rate to its bound on the null hypothesis and can identify local alternative hypothesis separate from null at a much quicker rate than the first.

[25,26,27] used the Cobb-Douglas and Exponential production functions to compare the efficiency of some tests for heteroscedasticity and found that Glejser and Park tests are the most powerful among other tests such as Breusch-Pagan, Goldfeld-Quandt and White.

[28] proposed a hypothesis test for heteroscedasticity in a nonparametric regression model; the test statistic is based on levene’s test. The distribution of the test statistic is based on the null hypothesis of homoscedasticity and local alternatives. Outcome of the simulations recommends that the proposed test is more efficient in some conditions mostly when the variance is a nonlinear function of the predictor.

2. Material and Methods

2.1. Estimation of Nonlinear Models

Some of the methods that can be used for estimating nonlinear models include Gauss-Newton [2,29], Newton- Ralpson [29,30], Steepest Descent [30,31, Marquardt's compromise [32] and Levenberg-Marquardt Algorithm [33,34]. The procedure of the Newton-Raphson method are as follows:
The Newton-Raphson Method [29,30], consider the model

$$Y_i = f_i\left(X,\theta\right) + u_i \qquad 1$$

Having all the assumptions of least square estimation which involves the minimization of the error terms

$$S\left(\theta\right) = \sum_{i=1}^{n} u_i^2 = \sum_{i=1}^{n}\left[Y_i - f\left(X,\theta\right)\right]^2 \qquad 2$$

In order to obtain the J - parameters $\theta_1, \theta_2, \ldots, \theta_J$, differentiate $S\left(\theta\right)$ w.r.t. θ and equate the result to zero to obtain J - normal equations.
that is,

$$S' = \frac{\partial S\left(\theta\right)}{\partial\theta} = 0 \qquad 3$$

The solution of θ is obtained by iteration. For the r^{th} iteration, the value of S' given θ is of the form

$$S_r' = 0 \qquad 4$$

Applying Taylor's expansion to the equation $S_r^{'}$ in the neighbourhood of θ_r of θ, (4) becomes

$$S_r' + \left(S_r''\right)\left(\theta - \theta_r\right) = 0 \qquad 5$$

where,

$$S_r' = -2\sum_{i=1}^{n}\left\{\left[Y_i - f_i\left(X,\theta\right)\right]\left[\frac{\partial f_i}{\partial\theta}\left(X,\theta\right)\right]\right\}_{|\theta=\theta_r} \qquad 6$$

and,

$$S_r'' = -2\sum_{i=1}^{n}\left\{\left[Y_i - f_i\left(X,\theta\right)\right]\left[\frac{\partial^2 f_i}{\partial\theta}\right] - \left[\frac{\partial f_i}{\partial\theta}\left(X,\theta\right)\right]'\left[\frac{\partial f_i}{\partial\theta}\left(X,\theta\right)\right]\right\}_{|\theta=\theta_r} \qquad 7$$

Observe that if

$u_i = Y_i - f_i\left(X,\theta\right)$ is the residual for a given θ, then 8

$$\frac{\partial^2 u_i}{\partial\theta^2} = \frac{\partial^2 f_i}{\partial\theta^2}\left(x,\theta\right) \qquad 9$$

So that (7) is written as

$$S_r'' = 2\sum_{i=1}^{n}\left\{u_i\frac{\partial^2 u_i}{\partial\theta^2} + \left[\frac{\partial f_i}{\partial\theta}(X,\theta)\right]'\left[\frac{\partial f_i}{\partial\theta}(X,\theta)\right]\right\}_{|\theta=\theta_r} \quad 10$$

(8) and (9) in S_r'' , then (5) becomes

$$-2\sum_{i=1}^{n}\left[u_i\frac{\partial f_i}{\partial\theta}(X,\theta)\right] + 2\sum_{i=1}^{n}\left\{u_i\left(\frac{\partial^2 u_i}{\partial\theta^2}\right) + \left[\frac{\partial f_i}{\partial\theta}(X,\theta)\right]'\left[\frac{\partial f_i}{\partial\theta}(X,\theta)\right]\right\}(\theta-\theta_1)_{|\theta=\theta_r} = 0 \quad 11$$

(11) is expressed as

$$-\sum_{i=1}^{n}\left[u_i\frac{\partial f_i}{\partial\theta}(X,\theta)\right] + \sum_{i=1}^{n}\left\{u_i\left(\frac{\partial^2 u_i}{\partial\theta^2}\right) + \left[\frac{\partial f_i}{\partial\theta}(X,\theta)\right]'\left[\frac{\partial f_i}{\partial\theta}(X,\theta)\right]\right\}(\theta-\theta_r) \quad 12$$

For the J parameters, let

$$H_i(\theta) = \frac{\partial^2 u_i}{\partial\theta_l\partial\theta_k} = \begin{pmatrix} \frac{\partial^2 u_i}{\partial\theta_1^2} & \frac{\partial^2 u_i}{\partial\theta_1\partial\theta_2} & \cdots & \frac{\partial^2 u_i}{\partial\theta_1\partial\theta_j} \\ \frac{\partial^2 u_i}{\partial\theta_2\partial\theta_1} & \frac{\partial^2 u_i}{\partial\theta_2^2} & \cdots & \frac{\partial^2 u_i}{\partial\theta_2\partial\theta_j} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 u_i}{\partial\theta_j\partial\theta_1} & \frac{\partial^2 u_i}{\partial\theta_j\partial\theta_2} & \cdots & \frac{\partial^2 u_i}{\partial\theta_j^2} \end{pmatrix} \quad 13$$

which is $J \times J$ Hessian matrix of u

therefore,

$$\sum_{i=1}^{n}u_i\left(\frac{\partial^2 u_i}{\partial\theta_i\partial\theta_j} = u_1\begin{bmatrix} \frac{\partial^2 u_1}{\partial\theta_1^2} & \frac{\partial^2 u_1}{\partial\theta_2\partial\theta_1} & \cdots & \frac{\partial^2 u_1}{\partial\theta_j\partial\theta_1} \\ \frac{\partial^2 u_1}{\partial\theta_1\partial\theta_2} & \frac{\partial^2 u_1}{\partial\theta_2^2} & \cdots & \frac{\partial^2 u_1}{\partial\theta_j\partial\theta_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 u_1}{\partial\theta_1\partial\theta_j} & \frac{\partial^2 u_1}{\partial\theta_2\partial\theta_j} & \cdots & \frac{\partial^2 u_1}{\partial\theta_j^2} \end{bmatrix} + \ldots + u_i\begin{bmatrix} \frac{\partial^2 u_i}{\partial\theta_1^2} & \frac{\partial^2 u_i}{\partial\theta_2\partial\theta_1} & \cdots & \frac{\partial^2 u_i}{\partial\theta_j\partial\theta_1} \\ \frac{\partial^2 u_i}{\partial\theta_1\partial\theta_2} & \frac{\partial^2 u_i}{\partial\theta_2^2} & \cdots & \frac{\partial^2 u_i}{\partial\theta_j\partial\theta_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 u_i}{\partial\theta_1\partial\theta_j} & \frac{\partial^2 u_i}{\partial\theta_2\partial\theta_j} & \cdots & \frac{\partial^2 u_i}{\partial\theta_j^2} \end{bmatrix}\right) \quad 14$$

The quantity $\frac{\partial f_i}{\partial\theta}(X,\theta)$, for $i = 1, 2, \ldots, n$ is written as

$$\frac{\partial f_i}{\partial \theta}(X,\theta)=\begin{pmatrix} \frac{\partial f_1}{\partial \theta_1}(X,\theta) & \frac{\partial f_1}{\partial \theta_2}(X,\theta) & \cdots & \frac{\partial f_1}{\partial \theta_j}(X,\theta) \\ \frac{\partial f_2}{\partial \theta_1}(X,\theta) & \frac{\partial f_2}{\partial \theta_2}(X,\theta) & \cdots & \frac{\partial f_2}{\partial \theta_j}(X,\theta) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial \theta_1}(X,\theta) & \frac{\partial f_n}{\partial \theta_2}(X,\theta) & \cdots & \frac{\partial f_n}{\partial \theta_j}(X,\theta) \end{pmatrix} \quad 15$$

also, it is simple to justify that

$$\sum_{i=1}^{n} u_i \left[\frac{\partial f_i}{\partial \theta}(X,\theta)\right]=\left[\frac{\partial f_i}{\partial \theta}(X,\theta)\right]' u \quad 16$$

Hence, using (13), (14), (15) and (16) in (12), to get

$$\left[\frac{\partial f_i}{\partial \theta}(X,\theta)\right]' u+\left\{\sum_{i=1}^{n} H_n(\theta)u_i+\left[\frac{\partial f_i}{\partial \theta_l}(X,\theta)\right]'\left[\frac{\partial f_i}{\partial \theta_k}(X,\theta)\right]\right\}'\left[\frac{\partial f_i}{\partial \theta}(X,\theta)\right]' u$$

If $\hat{\theta}$ is written as the θ_{r+1} , the last equation, then, is of the form

$$\theta_{r+1}=\theta_r+\left[\sum_{i=1}^{n} H_i(\theta)u_i+Z'Z\right]^{-1} Z'u \quad 17$$

where,

$$Z=\frac{\partial f_i}{\partial \theta}(X,\theta) \text{ for } i=1,2,\ldots,n \quad 18$$

and

$$\sum_{i=1}^{n} H_n(\theta)u_i=\sum_{i=1}^{n} u_i\left(\frac{\partial^2 u_i}{\partial \theta_i^2}\right)$$

2.2 The nonlinear econometric models investigated in this study are:

i. The Cobb-Douglass (CD) Model with multiplicative random error is written as

$$Y_i=\theta_1 K^{\theta_2} L^{\theta_3} e^{u_i} \quad 19$$

where,

Y_i is a vector of the dependent variables,

θ_1 is the intercept ,

θ_2 and θ_3 are the regression coefficients,

K is the Capital

L is the Labour
u_i is the random error.
Equation 19 is made linear by taking natural logarithms of the two sides of the equation. Thus, the following model

$$\ln(Y_i) = \ln\theta_1 + \theta_2 \ln K + \theta_3 \ln L + u_i$$

20
This is expressed as

$$Y_i^* = \theta_1^* + \theta_2 K^* + \theta_3 L^* + u_i$$

21
where,

$$\ln(Y_i) = Y_i^*, \ln\theta_1 = \theta^*, \ln K = K^*, \ln L = L^*$$

ii. The Constant Elasticity of Substitution (CES) model with multiplicative random error is written as

$$Y_i = \theta_1\left[\theta_2 K^{-\theta_3} + (1-\theta_2)L^{-\theta_3}\right]^{-\frac{1}{\theta_3}} e^{u_i}$$

22
Where
Y_i is a vector of the dependent variables,
θ_1 is the intercept ,
θ_2 and θ_3 are the regression coefficients,
K is the Capital
L is the Labour
u_i is the random error.
By applying the natural logarithms to the two sides of (22), gives

$$\ln(Y_i) = \ln(\theta_1) - \frac{1}{\theta_3}\ln\left[\theta_2 K^{-\theta_3} + (1-\theta_2)L^{-\theta_3}\right] + u_i \qquad 23$$

The result in (23) is intrinsically nonlinear since taking logarithms will not make the nonlinear function linear in parameters.
The model in (23) is linearised using [35] linearisation approach. Besides, a linear Taylor Series expansion around $\theta_3 = 0$ produced an intrinsically linear model.

$$\ln(Y_i) = \ln\theta_1 + \theta_2 \ln K + (1-\theta_2)\ln L - \frac{1}{2}\theta_3\left[\theta_2(1-\theta_2)(\ln K - \ln L)^2\right] + u_i$$
$$= \ln\theta_1 + \theta_2 \ln K + (1-\theta_2)\ln L + \theta_3\theta_2(1-\theta_2)\left[-\frac{1}{2}(\ln K - \ln L)^2\right] + u_i \qquad 24$$

2.3 Proposed Modified Breusch Pagan Test

Given the linear regression model of the k-variable

$$Y_i = \beta_1 + \beta_{21}X_{21} + \ldots + \beta_{k1} + u_1 \qquad 25$$

Provided the error variance σ_i^2 is represented as:

$$\sigma_i^2 = f(Z_{1i}, Z_{2i}, \ldots, Z_{mi}) \qquad 26$$

viz,
σ_i^2 is some function of the nonstochastic variable Z's; X's can serve as Z's.
It also assumed

$$\sigma_i^2 = \alpha_1 + \alpha_2 Z_{21} + \ldots + \alpha_m Z_{m1} \qquad 27$$

That is,

σ_i^2 is a linear function of the Z's. If $\alpha_2 = \alpha_3 = ... = \alpha_m = 0$, 28

one gets $\sigma_i^2 = \alpha_1$ which is constant.

That is, the main concept of the Breuch Pagan Test.

Thus, testing if σ_i^2 's are heteroscedastic, the hypothesis can be tested that $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq ... \neq \alpha_m \neq 0$ [for at least 1 αs are not the same]

Estimating the model in equation (27) to obtain the errors using least Square approach

Then the mean of least squared residual

$$\hat{\sigma}^2 = \frac{n\sum u_i^2}{n-k} \quad \text{are estimated} \qquad 29$$

Compute the variable

$$p_i = \frac{\hat{u}_i^2}{\hat{\sigma}_i^2}$$

By regressing the variable p_i against the Z's as

$$p_i = \alpha_1 + \alpha_2 Z_{21} + ... + \alpha_m Z_{m1} \qquad 30$$

obtain the ESS

Lastly, the test statistic T is computed

$$T = ESS/2$$

From the assumption of normality and homoscedasticity,

$$T \sim \chi^2_{m-1}$$

Assuming T value is higher than the critical value, then there is presence of heteroscedasticity in the data.

2.4 Criteria for Comparison

2.4.1 P- Value

The P value is the probability of finding the observed or more extreme results when the null hypothesis (H_0) of a study question is true. It is used in the context of null hypothesis testing in order to quantify the idea of statistical significance of the evidence. It also described in terms of rejecting H_0 when it is actually true, however, it is not a direct probability of this state. The null hypothesis (H_0) is usually an hypothesis of 'no difference'. The smaller the P value, the higher the significance because it shows that the hypothesis under consideration may not adequately explain the observation.

2.4.2 Type I and Type II Error

A type I error occurs when the null hypothesis (H_0) is true but is rejected. A type I error may be likened to a so-called false positive (a result that indicates that a given condition is present when it actually not present).

The type I error rate or significance level is the probability of rejecting the null hypothesis (H_0) given that it is true and it is denoted by α .

A type II error occurs when the null hypothesis (H_0) is false but erroneously fails to be rejected. A type II error may be compared with so-called false negative. This type of error is committed when one fails to believe a true alternative hypothesis.

The type II rate is the probability of accepting null hypothesis (H_0) given that it is false and it is denoted by β .

Table 1 Table of Error Types

<table>
<tr><th colspan="2" rowspan="2">ERROR TYPES</th><th colspan="2">NULL HYPOTHESIS (H_0)</th></tr>
<tr><th>TRUE</th><th>FALSE</th></tr>
<tr><td rowspan="2">Decision about Null Hypothesis (H_0)</td><td>Fail to reject</td><td>Correct Inference
(True Negative)
Probability = $1-\alpha$</td><td>Type II Error
(False Negative)
Probability = β</td></tr>
<tr><td>Reject</td><td>Type I Error
(False Positive)
Probability = α</td><td>Correct Inference
(True Positive)
Probability = $1-\beta$</td></tr>
</table>

2.4.3 Power of a Test

This is the probability that the test rejects the null hypothesis (H_0) when a specific alternative hypothesis (H_1) is true. The statistical power ranges from 0 to 1 and as statistical power increases, the probability of making a type II error (wrongly failing to reject the null hypothesis (H_0) decreases). For a type II error, probability of β, the corresponding power is $1-\beta$.

It can be equivalently thought of as the probability of accepting the alternative hypothesis (H_1) when it is true, that is, the ability of a test to detect a specific effect, if that specific effect actually exists. That is,

$$\alpha = \Pr(type\ I\ error)$$

$$= \Pr(rejecting\ H_0 / H_1\ is\ true)$$

$$\beta = \Pr(type\ II\ error)$$

$$= \Pr(accepting\ H_0 / H_1\ is\ true)$$

$$= 1 - \Pr(rejecting\ H_0 / H_1\ is\ true)$$

Therefore,

$$\Pr(rejecting\ H_0 / H_1\ is\ true) = 1 - \beta = Power$$

Power = $\Pr(rejecting\ H_0 / H_1\ is\ true) = 1 - \beta$

Cohen (1998) proposed that the maximum accepted probability of a type II error should be 20%, that is, $\beta = 0.2$.

The power of the tests could be assessed using,

$$\pi = 1 - \beta$$

$\pi = 1 - 0.2 = 0.8$, as a standard of adequacy.

As the power increases, there is a decreasing probability of a type II error which also referred to as false negative rate (β) since the power of a test is equal to $1-\beta$. A similar concept is the type I error probability, also referred to as the false positive rate or the level of significance of a test under the null hypothesis (H_0).

2.5 Error Variance Structures and Tests for Heteroscedasticity

The study assumed multiplicative form of heteroscedasticity error structure model discussed by [36]. The equation is given as

$$Y_i = X_i\beta + u_i \quad (31)$$

where

$$u_i \sim N\left(0, \sigma_i^2\right)$$

$$\begin{aligned}\sigma_i^2 &= \sigma^2 E\left(y_i\right)^2 \\ &= \sigma^2\left(\beta_1 x_{i1} + \cdots + \beta_k x_{ik}\right)^2 \\ &= \sigma^2 \exp\left(q, \lambda\right) \end{aligned} \quad (32)$$

For $i = 1, 2, \ldots, n$ where Y_i is the i^{th} observation,

X and q are the $I \times k$ and $I \times \left(J - I\right)$ vectors of independent variables respectively.

β and λ are the vectors of unknown parameters. λ is a real constant determining heteroscedasticity levels.

1.6 Data Generating Procedures

i. Explanatory variables X_1 and X_2 were uniformly distributed in a model infected heteroscedasticity proposed by [36] to obtain response variable Y. The parameter values θ_1 =0.2, θ_2 =0.5 and θ_3 =0.3 for the three models were arbitrary chosen. For the simulation, 10 and 30, 50 and 100, 150 and 200 were the sample sizes selected and each was respectively grouped into small, medium and large sample units before being replicated 10,000 times.

ii. The OLS was used to estimate the parameters of the transformed models and hence the residual of the model. The levels or degree of heteroscedasticity, λ, introduced were 0.1 for mild, 0.5 for moderate and 0.9 for severe.

iii. Empirical study was carried on the data extracted from World Development Indicator (WDI) database for a period of Thirty years, spanning from 1990-2019. The Gross Domestic Product (GDP) represents response variable, Gross Fixed Capital Formation (GFCF) and Total Labour Force (TLF) are the two explanatory variables.

iv. The level of significance used were 0.01 and 0.05

II. Analysis and Discussion of Results

The results obtained from the simulated and empirical data sets for the power of the tests viz Modified Breusch Pagan (MBP), Breusch Pagan (BP), Parck and White with the nonlinear models: CD and CES were presented. The levels of heteroscedasticity λ introduced were 0.1 for mild, 0.5 for moderate and 0.9 for severe while 10 and 30, 50 and 100, 150 and 200 were the sample sizes selected and each was respectively grouped into small, medium and large sample units with α =0.001 and 0.05 were produced in the tables.

Presentation of Results Obtained from Simulated Data Sets.

Table 2: Power of the Tests for CD Model at $\alpha = 0.01$

λ	TEST	SAMPLE SIZE				
		30	50	100	150	200
0.1	**BREUSCH-PAGAN**	0.255700	0.872000	0.615000	0.542800	0.351000
	WHITE	0.287300	0.121500	0.615100	0.336300	0.122100
	GOLDFELD QUANDT	-	-	-	0.000000	0.000002
	MBP	0.976794	0.940744	0.961384	0.987918	0.982622
0.5	**BREUSCH-PAGAN**	0.380400	0.695200	0.403100	0.532900	0.275000
	WHITE	0.506600	0.530500	0.454400	0.513900	0.883500
	GOLDFELD QUANDT	-	-	0.208280	0.009584	0.000179
	MBP	0.979504	0.991991	0.988732	0.982641	0.986620
0.9	**BREUSCH-PAGAN**	0.819000	0.841900	0.024400	0.815600	0.015700
	WHITE	0.287300	0.989100	0.142600	0.121400	0.507200
	GOLDFELD QUANDT	-	0.406418	0.016276	0.026878	0.000213
	MBP	0.995100	0.949792	0.950798	0.996576	0.997348

Table 2 above reports the result of the simulated data for the CD model at $\alpha = 0.01$. As shown in the given table, each of the sample units, at different levels of heteroscedasticity, the power of the tests for MBP was increasing systematically. It was also observed that at moderate level of heteroscedasticity, White strength had a significant improvement. There was an improvement in all the tests when λ =0.5 and 0.9 at medium sample. Conclusively, the Goldfeld and Quandt test is uncertain at different levels of heteroscedasticity and sample sizes considered.

Table 3: Power of the Tests for CD Model at $\alpha = 0.05$

λ	TEST	SAMPLE SIZE					
		10	30	50	100	150	200
0.1	**BREUSCH-PAGAN**	-	0.488000	0.650800	0.022400	0.929900	0.770400
	WHITE	-	0.388400	0.924900	0.274200	0.885500	0.171100
	GOLDFELD QUANDT	-	-	-	0.045789	0.001613	0.000242
	MBP	1.000000	0.976794	0.940744	0.935572	0.998622	0.996401
0.5	**BREUSCH-PAGAN**	**-**	0.934500	0.325200	0.071800	0.359300	0.174600
	WHITE	0.223100	0.287300	0.571700	0.209700	0.611300	0.334900
	GOLDFELD QUANDT	**-**	-	0.230167	0.009457	0.000460	0.002074
	MBP	1.000000	0.984533	0.930281	0.998854	0.945151	0.998436
0.9	**BREUSCH-PAGAN**	**-**	0.903000	0.488500	0.167000	0.508500	0.757100
	WHITE	0.223100	0.287300	0.312900	0.294400	0.888400	0.107400
	GOLDFELD QUANDT	**-**	-	0.279128	0.001565	0.001927	0.001174
	MBP	1.000000	0.995100	0.949792	0.950798	0.989501	0.997401

Table 3 above shows the outcome of the simulated data for the CD Model at $\alpha = 0.05$. It is observed that as the sample unit progressively increased at different level of heteroscedasticity, the power for MBP remain powerful. Furthermore, it shows that at $\lambda = 0.1$ when sample size is 150 all the powers in each of the test besides the Goldfeld Quandt test improved. Similar results were obtained at $\lambda = 0.9$ when sample size is 150.

Table 4: Power of the Tests for CES Model at $\alpha = 0.01$

λ	TEST	SAMPLE SIZE				
		30	**50**	**100**	**150**	**200**
0.1	**BREUSCH-PAGAN**	0.501100	0.365100	0.562500	0.456200	0.594100
	WHITE	0.287300	0.136800	0.014600	0.102200	0.305700
	GOLDFELD QUANDT	-	0.042116	0.398248	0.191242	0.164664
	MBP	0.947663	0.998790	0.930048	0.996403	0.998683
0.5	**BREUSCH-PAGAN**	0.291600	0.198100	0.307400	0.002600	0.915900
	WHITE	0.306200	0.488900	0.245100	0.250000	0.428000
	GOLDFELD QUANDT	-	0.000246	0.122650	0.000000	0.000092
	MBP	0.914696	0.991438	0.941818	0.994757	0.999270
0.9	**BREUSCH-PAGAN**	0.910300	0.576900	0.379400	0.478700	0.187900
	WHITE	0.287300	0.323400	0.520800	0.349200	0.000000
	GOLDFELD QUANDT	-	0.248282	0.002210	0.036179	0.000041
	MBP	0.952222	0.955736	0.989762	0.999564	0.995094

Table 4 shows the simulation result values for CES Model at $\alpha = 0.01$. Here the result indicates that the rate at which each sample size selected increased significantly influenced the level of heteroscedasticity that takes place at every level. Thus, confirming that the power of MBP remains powerful. Breusch-Pagan,White and Goldfeld Quandt tests have a very low power in detecting the presence of heteroscedasticity at every level

Table 5: Power of the Tests for CES Model at $\alpha = 0.05$

λ	TEST	SAMPLE SIZE					
		10	**30**	**50**	**100**	**150**	**200**
0.1	**BREUSCH-PAGAN**	-	0.501100	0.160400	0.374400	0.910300	0.223300
	WHITE	0.223100	0.287300	0.072600	0.520400	0.225500	0.079700
	GOLDFELD QUANDT	-	-	0.011362	0.002220	0.059527	0.105362
	MBP	1.000000	0.947663	0.964533	0.976460	0.990481	0.982261
0.5	**BREUSCH-PAGAN**	**-**	0.687300	0.576600	0.376900	0.480900	0.187700
	WHITE	0.223100	0.287300	0.322700	0.520600	0.346600	0.000000
	GOLDFELD QUANDT	**-**	-	0.190341	0.002203	0.036445	0.000407
	MBP	0.995208	1.000000	0.974836	0.930728	0.994404	0.999063
0.9	**BREUSCH-PAGAN**	**-**	0.910300	0.576900	0.379400	0.478700	0.006700
	WHITE	0.223100	0.287300	0.323400	0.520800	0.349200	0.000000
	GOLDFELD QUANDT	**-**	-	0.190689	0.002210	0.036179	0.000000
	MBP	1.000000	0.928579	0.914372	0.992718	0.996304	0.998667

Table 5 presents the result of simulation values for CES at $\alpha = 0.05$. It reveals that at each point where heteroscedasticity occurred there was a symmetric relation with the increment in sample size with the overall result being that the values obtained in the tests for MBP remained forceful. The strength of the test for Park improved from small to larger sample sizes. Breusch Pagan test has low power while the White and Goldfeld Quandt tests are weaker in detecting the presence of heteroscedasticity at every level.

Presentation of Results Obtained from Real Life Data Sets

Table 6: Original Data Extracted from World Development Indicator ('000s)

YEAR	GDP	CAPITAL	LABOUR
1990	281550270	40121310	29591
1991	329070750	45190230	30343
1992	555445510	70809160	31110
1993	715241870	96915510	31890
1994	945557020	105575490	32702
1995	2008564010	141920240	33551
1996	2799036110	204047610	34359
1997	2906624900	242899790	35214
1998	2816406010	242256260	36108
1999	3312240870	231661690	37044
2000	4717332100	331056730	37986
2001	4909526480	372135650	38933
2002	7128203100	499681530	39926
2003	8742646650	865876460	40907
2004	11673602240	863072620	41748
2005	14735323980	804400820	42854
2006	18709786480	1546525650	43908
2007	20940910900	1936958210	45036
2008	24665244300	2053005940	46231
2009	25236056300	3050575920	47480
2010	55469350300	9183059440	48781
2011	63713359400	9897197180	50069
2012	72599630000	10281951750	51416
2013	81009964600	11478080090	52824
2014	90136985000	13593779780	54261
2015	95177735684	14112170000	55789
2016	102575418034	15076795270	57352
2017	113711630000	16908133137	57856
2018	127736830000	24550243168	60517
2019	144210490000	37015484902	63227

Source: Data from Database; Word Development Indicator (WDI).

Table 6 shows the Output, that is, the Gross Domestic Product (GDP) representing Income and the Inputs which are Gross Fixed Capital Formation (GFCF) representing Capital and Total Labour Force (TLF) representing Labour in Nigeria.

Figure 1: Scatter plot of response and explanatory variables

This graph shows the systematic pattern of the relationship between income and the two explanatory variables (Capital and Labour). The pattern depicts the presence of heteroscadasticity because there are outliers in the variables of interest. Some observations deviate from the linear correlation existed in the graph.

Table 7: The Power of Heteroscedasticity Tests for the three Models for $\alpha = 0.01$ and 0.05

TESTS	MODELS		
	CD	CES	EXPONENTIAL
BREUSCH-PAGAN	0.2253	0.0290	0.6424
WHITE	0.1125	0.1218	0.3520
GOLDFELD QUANDT	0.9658	0.9307	0.9726
MBP	0.9944	0.9923	0.9952

The result in Table 7 shows that MBP and Goldfeld-Quandt tests are the most powerful tests while Breusch-Pagan and White test are the least powerful test for all the models. The same result when $\alpha = 0.05$

III. Conclusion

The two functional forms of production function namely: CD and CES were transformed to make them linear in parameters, that is, intrinsically linear models. The four tests for heteroscedasticity were also considered, these are Breusch-Pagan, Goldfeld-Quandt , White and MBP. The real life data on GDP in Nigeria for the period of twenty-seven (27) years, that is, 1990-2016, extracted from World Development Indicator (WDI) database was used for gross domestic product (GDP), capital (K) and Labour (L). The simulated and real life data were used for the models and tests for comparison. From the analysis, it can be concluded that the MBP tests performed excellently and was

discovered to be the most powerful test in detecting the presence of heteroscedasticity; at all levels of heteroscedasticity, for all sample sizes and models both for the simulated and real life data among the classes of other tests considered. Further results showed that, when the level of heteroscedasticity increased, the power of some of the tests also increased. Also, in determining the effect of size of the sample on the power of the tests, the result showed that when the size of the sample increased, the power of some of the tests also increased.
Finally, the existing Breush Pagan test was modified to improve its efficiency. The results showed that the proposed modified test performed creditably well via-a-vis all of the existing methods considered under the two models at varying parametric combinations. The superiority of the modified Breush Pagan test method was established under extensive simulated environment and the results were validated on real life data set.

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APPENDIX

Results of Breusch-Pagan Test Using Simulated Data

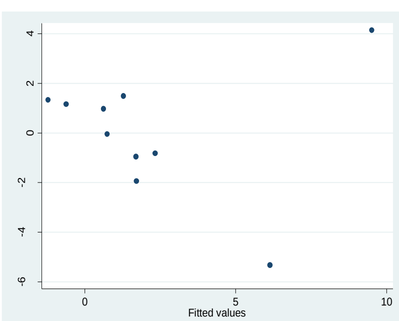
Showing Presence of Heteroscedasticity

Sample Size (n)	Level of Heteroscedasticity (λ)	BP test statistic	P value	Remark
10	0.1 0.5 0.9	7.85 6.49 7.53	0.0051 0.0108 0.0050	Heteroscedasticity Heteroscedasticity Heteroscedasticity
30	0.1 0.5 0.9	38.33 27.50 6.85	0.0000 0.0000 0.0089	Heteroscedasticity Heteroscedasticity Heteroscedasticity
50	0.1 0.5 0.9	10.82 18.59 20.69	0.0010 0.0000 0.0000	Heteroscedasticity Heteroscedasticity Heteroscedasticity
100	0.1 0.5 0.9	9.27 8.32 9.42	0.0020 0.0039 0.0021	Heteroscedasticity Heteroscedasticity Heteroscedasticity
150	0.1 0.5 0.9	398.54 399.53 395.82	0.0000 0.0000 0.0000	Heteroscedasticity Heteroscedasticity Heteroscedasticity
200	0.1 0.5 0.9	526.96 526.63 525.14	0.0000 0.0000 0.0000	Heteroscedasticity Heteroscedasticity Heteroscedasticity

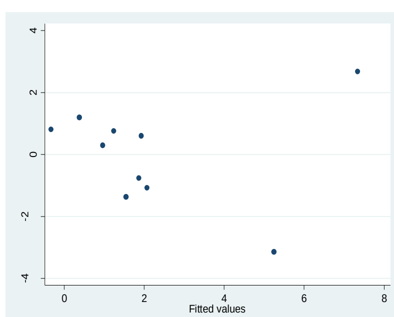
Graph of Residual Plots of Breusch-Pagan Test Using Simulated Data

Showing Presence of Heteroscedasticity

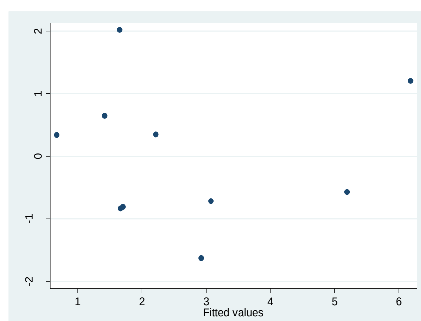
N=10 $\lambda = 0.1$



N=10 $\lambda = 0.5$



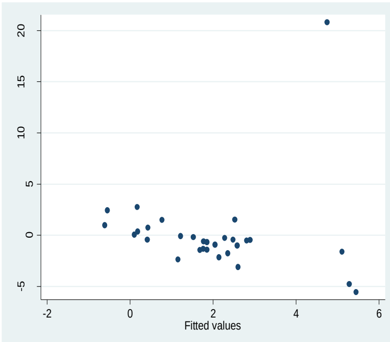
N=10 $\lambda = 0.9$

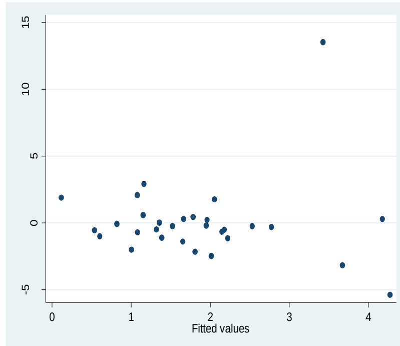


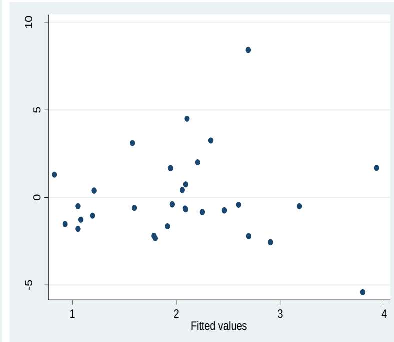
N=30 $\lambda = 0.1$

N=30 $\lambda = 0.5$

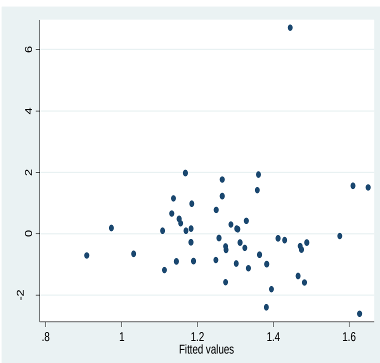
N=30 $\lambda = 0.9$

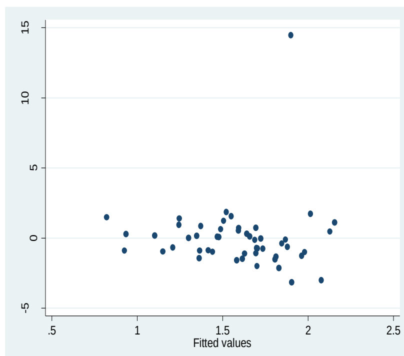


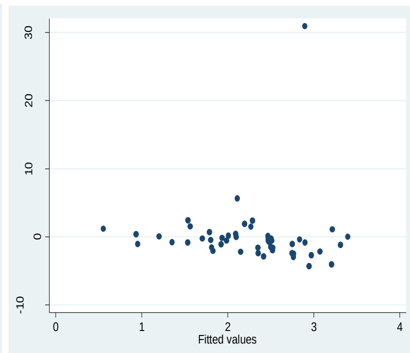




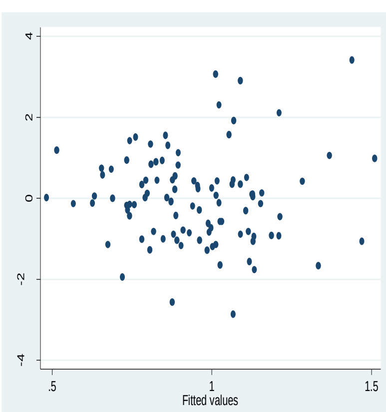
N=50 $\lambda = 0.1$ N=50 $\lambda = 0.5$ N=50 $\lambda = 0.9$

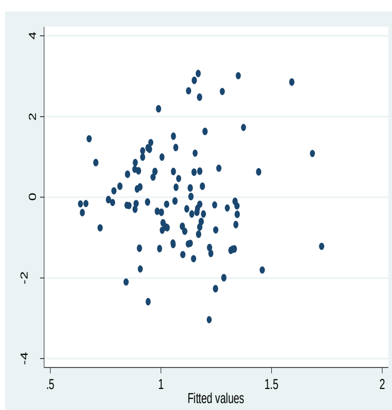


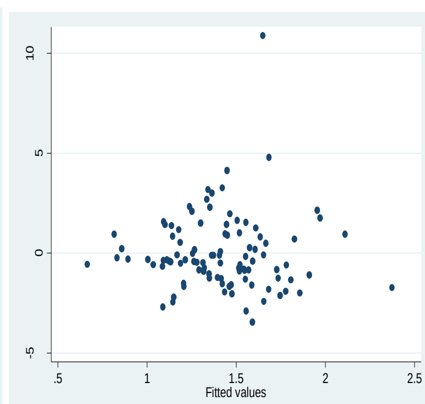




N=100 $\lambda = 0.1$ N=100 $\lambda = 0.5$ N=100 $\lambda = 0.9$







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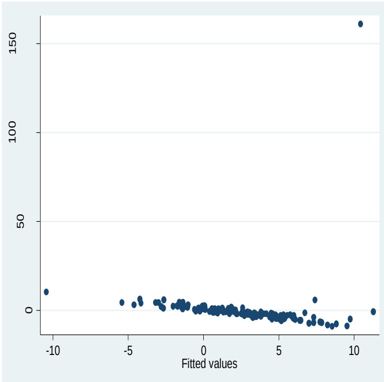

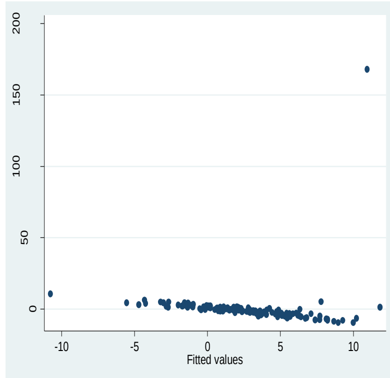

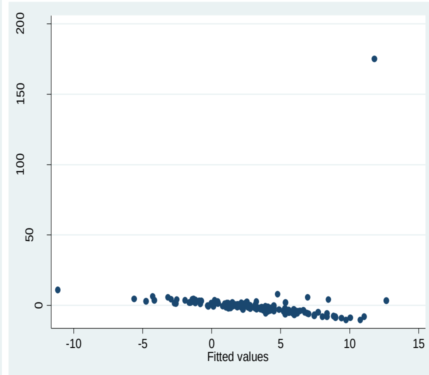

N=200 $\lambda = 0.1$ N=200 $\lambda = 0.5$ N=200 $\lambda = 0.9$