A SELF-ADAPTIVE INERTIAL EXTRAGRADIENT ALGORITHM FOR SOLVING NON-LIPSCHITZ VARIATIONAL INEQUALITIES

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ABSTRACT
In this paper, we introduce a modified extragradient algorithm for solving variational inequality problems involving pseudomontone with non-Lipschitz continuous operators. The algorithm which combines the inertial technique and the extragradient algorithm, we show the algorithm is globally convergence.

Keywords: *Variational inequality problem, Extragradient method, Inertial-type algorithm, Non-Lipschitz continuity, Convergence.*

1. INTRODUCTION

In this paper, we consider the following classical variational problem, denoted by VIP(F, C), which is to find a point x^* such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

Where $F: R^m \to R^m$ is an operator, $C \in R^m$ is a nonempty closed convex set and $\langle \cdot,\cdot \rangle$ denotes the innerproduct in R^m. The solution set of VIP(F, C) is denoted by C^*. Variational inequalities are an important class of nonlinear problems, which have applications in many aspects. For example, operational research problems, equilibrium problems in the economic field, and urban transportation network modeling. It not only unifies the concepts in applied mathematics, but also strengthens the knowledge system of complementary problems, optimization problems and equilibrium problems [1, 2, 3]. With more and more scholars study it, there are many methods for solving the variational inequality problem (1.1) and its variants [4, 5, 6, 7, 8, 9, 10]. The simplest one is the following method, which defined as gradient projection method [11, 12],

$$x_{n+1} = P_C(x_n - \lambda F(x_n)),$$

Where $\lambda \in (0, \frac{2\eta}{L^2})$, F is η-strongly monotone L-Lipschitz continuous on C. It is known that the assumption of this algorithm is very restrictive. In order to overcome the above problems, some scholars have proposed a new method[double projection method], which defined as extragradient projection method[11, 12], the specific algorithm is as follows[13]:

$$\begin{cases} y_n = P_C(x_n - \lambda F(x_n)) \\ x_{n+1} = P_C(x_n - \lambda F(y_n)) \end{cases},$$

Where $\lambda \epsilon (0, \frac{1}{L})$, the operator F is monotone and Lipschitz continuous. It is worth noting that the stepsize in the above methods depend on the Lipschitz constant, in the case when the operator F is not Lipschitz continuous or the Lipschitz constant L is very difficult to compute, these methods are not applicable. Recently, the EGM has been interested and developed under suitable conditions [14, 15, 7, 16, 17, 18]. Iusem [19] proposed a new algorithm that its convergence is guaranteed without the Lipschitz continuity, the algorithm's stepsize rule is as follow: $\lambda_n = \lambda^{l^{j_n}}$, where j_n is the smallest non-negative integer j satisfying $\lambda^{l^j} \|F(x_n) - F(y_n)\| \leq \mu \|x_n - y_n\|$; in particular, Trinh

[20] introduced a new algorithm which without the Lipschitz continuity of operator(see Algorithm 1) for solving the pseudomonotone VIP as follows:

Algorithm 1

Initialization Choose arbitrarily $x^{-1}, x^0, y^0 \in C; \rho, \sigma \in (0,1), \lambda_{-1}, \alpha_{-1} \in (0, \infty), set\ n := 0.$

Step 1 Given $\lambda_{n-1}, x_{n-1}, y_n\ and\ x_n$.

If $\lambda_{n-1}\|F(x^{n-1}) - F(y^n)\| \leq \rho\|x^{n-1} - y^n\|$ then set $\lambda_n = \lambda_{n-1}$, else set $\alpha_n = \alpha_{n-1}\delta$.

Compute

$$\lambda_n = \frac{\alpha_n}{max\{1, F(x^n)^2\}},$$

$$y_{n+1} = P_C\big(x_n - \lambda_n F(x_n)\big),$$

$$x_{n+1} = P_C\big(x_n - \lambda_n F(y_{n+1})\big).$$

Step 2 If $y_{n+1} = x_n$, then stop, else update n := n + 1 and go to Step 1.

Note that, different from [19], this algorithm does not need to calculate values of the operator F many times at each iteration and not require its step-sizes tending to zero. Therefore, the algorithm reduces the amount of calculation and time consumed, it requires the mapping F being pseudomonotone and Lipschitz continuous only on the feasible set instead of on the whole space.

It is also known that the inertial-type algorithm is one of the effective methods for speeding up the convergence properties of fundamental algorithms, see [21]. The main feature of inertial algorithms is that the next iterate is constructed from the two previous iterates. Motivated and inspired by inertial-type method [22] and above algorithms [20], we will introduce a kind of inertial extragradient algorithm. In this paper, our goal is to obtain convergence result for variational inequalities by adding an inertial parameter.

The structure of this paper is as follows. In Section 2, we first introduce some concepts and preliminary results used in this paper. Section 3 propose an inertial extragradient algorithm and analyse the convergence of it.

2. PRELIMINARIES

In this section, we introduce some concepts and lemmas that will be used. Throughout the paper, let $C \in R^m$ is a nonempty closed convex set and the operator $F: C \to R^m$ is pseudomonotone and non-Lipschitz continuity.

In this paper, the orthogonal projection of x onto C is denoted by $P_C(x)$ such that

$$P_C(x) = argmin_{y \in C}\|y - x\|.$$

For all λ > 0, $x^*$is a solution of (1.1) if and only if

$$x^* - P_C(x^*) = 0.$$

Definition 2.1 Let $F: C \to R^m$ be a mapping, then

(1) F is monotone on C if for all x, y ∈ C, we have

$$\langle F(x) - F(y), x - y \rangle \geq 0.$$

(2) F is pseudomonotone on C, if for all x, y ∈ C, we have

$$\langle F(x), y - x\rangle \geq 0 \Rightarrow \langle F(y), y - x\rangle \geq 0.$$

(3) F is γ-strongly pseudomonotone on C, if there exists a constant γ > 0, for all x, y ∈ C, we have

$$\langle F(x), y - x\rangle \geq 0 \Rightarrow \langle F(y), y - x\rangle \geq \gamma\|x - y\|^2.$$

(4) F is called L-Lipschitz continuous, if there exists a constant L > 0, for all x, y ∈ C, we have

$$\|F(x) - F(y)\| \leq L\|x - y\|.$$

It is easy to see that in the above definitions, (3) is included in (2), (1) is included in (2), but the converse is not true. This paper will use following lemmas:

Lemma 2.1[23] Let C be a nonempty, closed and convex subset of R^m, for ∀x ∈ R^m. Then

(1) $\langle P_C(x) - x, y - P_C(x)\rangle \geq 0, \forall y \in C;$

(2) $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle, \forall y \in R^m;$

(3) If y ∈ C, then $\|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|x - P_C(x)\|^2$.

Lemma 2.2[22] Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences in $[0, \infty)$ such that

$$a_{n+1} \leq a_n + b_n(a_n - a_{n-1}) + c_n, \forall n \geq 1, \sum_{n=1}^{\infty} c_n < \infty,$$

and there exists a real number b with 0 ⩽ bn ⩽ b < 1 for all n ∈ N. Then the following results hold:

(1) $\sum_{n=1}^{\infty}[a_n - a_{n-1}]_+ < \infty$, where [t]+ := max{t, 0};

(2) there exists $a^* \in [0, \infty)$ such that $\lim_{n\to\infty} a_n = a^*$.

Lemma 2.3[20] Suppose that the mapping F : C → R^n is continuous. Then, for all bounded sequences $\{x_n\}, \{y_n\} \subset C$ satisfying $\|x_n - y_n\| \to 0$, it holds that

$$\|F(x_n) - F(y_n)\| \to 0.$$

3. MAIN RESULTS

In this section, we discuss a modified algorithm which is called self-adaptive inertial extragradient algorithm. Based on Algorithm1, our algorithm add an inertial term. Below we give the conditions that will be used in the proof process.

Condition 1 C is a nonempty, closed and convex set. We always assume that the solution set of the variational inequality is nonempty, i.e.,$x^* \in C^*$.

Condition 2 The operator F is pseudomonotone on C.

Condition 3 The operator F is continuous on C, and for any sequences $\{x_n\}$, $\{y_n\}$ satisfying $\|x_n - y_n\| \to 0$, there exists a constant m > 0 such that $\|F(x_n) - F(y_n)\| \leq m$ for all n $\geqslant$ 0.

When the Condition1-Condition3 hold, we show the algorithm as follows:

Algorithm 2

Initialization Choose arbitrarily $x^1, x^0, y^0 \in C; \rho, \sigma \in (0,1), \lambda_1, a, b \in (0, \infty), set\ n := 0$.

Step 1 Computer $z_n = x_n + b_n(x_n - x_{n-1})$. Where

$$b_n = \begin{cases} min\left\{\frac{1}{n^2\|x_n - x_{n-1}\|^2}, a\right\}, if\, \|x_n \neq x_{n-1}\|. \\ b, \qquad otherwise. \end{cases}$$

Step 2 Compute

$$y_n = P_C\big(z_n - \lambda_n F(z_n)\big)$$

If $y_n = z_n$, then stop and y_n is a solution of (1.1). Otherwise go to step3.

step 3 Compute

$$x_{n+1} = P_C\big(z_n - \lambda_n F(y_n)\big).$$

Update

$$\lambda_{n+1} = \frac{a_{n+1}}{max\{1, \|F(z_n)\|^2\}},$$

where

$$a_{n+1} = \begin{cases} a_n, if \;\; a_n\|F(z_n) - F(y_n)\| \leq \rho\|z_n - y_n\|; \\ a_n\delta, \qquad otherwise. \end{cases},$$

Set n := n + 1 and go to step 1.

Remark 3.1 According to definitions of $\{\lambda_n\}$ and $\{a_n\}$, the sequence $\{\lambda_n\}$ is nonincreasing. Next, to analyze the feature of the stepsize, we only need to analysis $\{a_n\}$.

Case 1 When there exists a constant M > 0, such that the sequence $\{a_n\} \to M$:

Lemma 3.1 Assume that Condition1 and Condition2 hold, let $\{z_n\}$, $\{x_n\}$, $\{y_n\}$ are three sequences generated by Algorithm 2, then when n → ∞, there have

$$\|z_n - x_n\| \to 0, \|y_n - x_n\| \to 0, \|z_n - y_n\| \to 0.$$

Proof: From the definition of b_n, we can know $b_n\|x_n - x_{n-1}\|^2 \le \frac{1}{n^2}$, so we have $\sum_{n=1}^{\infty} b_n\|z_n - y_n\|^2 < \infty$, which means that

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0,$$

consequently, we obtain

$$\|z_n - x_n\|^2 = {b_n}^2\|x_n - x_{n-1}\|^2 \to 0 \quad (when\ n \to \infty),$$

i.e.,

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.2}$$

By the definition of z_n and Lemma 2.1, for ∀α ∈ C, we obtain

$$\langle \alpha - y_n, z_n - \lambda_n F(y_n) - y_n\rangle \le 0,$$

i.e.,

$$\langle y_n - z_n, y_n - \alpha\rangle \le \lambda_n\langle F(z_n), \alpha - y_n\rangle. \tag{3.3}$$

Similarly, for ∀α ∈ C, we get

$$\langle x_{n+!} - z_n, x_{n+!} - \alpha\rangle \le \lambda_n\langle F(y_n), \alpha - x_{n+1}\rangle, \tag{3.4}$$

when $\alpha \in C^* \subset C$, let $\alpha = x^*$, from (3.3) and (3.4) we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|x_{n+1} - y_n + y_n - x^*\|^2\\ &= \|x_{n+1} - y_n\|^2 + \|y_n - z_n + z_n - x^*\|^2\\ &+2\langle x_{n+1} - y_n, y_n - x^*\rangle\\ &\le -\|x_{n+1} - y_n\|^2 + \|z_n - x^*\|^2 - \|y_n - z_n\|^2\\ &+2\lambda_n\langle F(z_n) - F(y_n), x_{n+1} - y_n\rangle + 2\lambda_n\langle F(y_n, x^* - y_n)\rangle\\ &\le -\|x_{n+1} - y_n\|^2 + \|z_n - x^*\|^2 - \|y_n - z_n\|^2\\ &+2\lambda_n\|F(z_n) - F(y_n)\|\|x_{n+1} - y_n\| + 2\lambda_n\langle F(y_n), x^* - y_n\rangle \end{aligned} \tag{3.5}$$

Since $a_n \to M > 0$, so there exists a real number k with k > 0 such that for ∀n ⩾ k we have

$$\lambda_n\|F(z_n) - F(y_n)\| < \rho\|y_n - z_n\|. \tag{3.6}$$

From (3.5) and (3.6), by the pseudomonotonicity of F we know that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le -\|x_{n+1} - y_n\|^2 + \|z_n - x^*\|^2 - \|y_n - z_n\|^2\\ &+2\rho\|z_n - y_n\|\|x_{n+1} - y_n\| + 2\lambda_n\langle F(y_n), x^* - y_n\rangle\\ &\le \|z_n - x^*\|^2 - (1-\rho)\|x_{n+1} - y_n\|^2 - (1-\rho)\|y_n - z_n\|^2, \end{aligned} \tag{3.7}$$

Letting i → ∞, by (3.2) we obtain

$$\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \ge (1-\rho)\|x_{n+1} - y_n\|^2 - (1-\rho)\|y_n - x_n\|^2,$$

therefore,

$$\lim_{n\to\infty}\|y_n - x_n\| = 0, \tag{3.8}$$

from (3.2) and (3.8), we obtain

$$\lim_{n\to\infty}\|y_n - z_n\| = 0.$$

Theorem 3.1 If Condition1 and Condition2 hold, then the sequence $\{x_n\}$ generated by Algorithm2 converges to a solution x^* of (1.1).

Proof: From (3.2) and (3.7) we know

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 - (1-\rho)\|x_{n+1} - y_n\|^2 - (1-\rho)\|y_n - x_n\|^2,$$

for ∀n ⩾ k, then sequence $\{\|x_n - x^*\|\}$ is nonincreasing, so $\{\|x_n - x^*\|\}$ is convergent. Hence, there exists subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to \bar{x} \in C$.

Now, for $\forall n_i \geqslant$ k, by (3.3) we have

$$\langle y_{n_i} - \alpha, y_{n_i} - z_{n_i}\rangle + \lambda_{n_i}\langle F(z_{n_i}), y_{n_i} - z_{n_i}\rangle \le \lambda_{n_i}\langle F(z_{n_i}), \alpha - z_{n_i}\rangle,$$

from the definition of λ_n, when $a_n \to M > 0$, there exists constant d with 0 < d ⩽ M such that λ_n→ d. Therefore, λ_n is bounded. Hence, from Lemma3.1 we know that

$$\langle F(x_{n_i}), \alpha - x_{n_i}\rangle \ge 0,$$

when i → ∞, we obtain

$$\langle F(\bar{x}), \alpha - \bar{x}\rangle \ge 0,$$

hence,

$$\lim_{n\to\infty}\|x_n - \bar{x}\| = \lim_{i\to\infty}\|x_{n_i} - \bar{x}\| = 0.$$

Consequently, $\bar{x} \in C^*$, the sequence $\{x_n\}$ is convergent.

Case 2 When the sequence $\{a_n\}$ → 0:

Lemma 3.2 Assume that all conditions hold, let $\{z_n\}$, $\{x_n\}$, $\{y_n\}$ are three sequences generated by Algorithm 2, then when n → ∞, there have

$$\|z_n - x_n\| \to 0, \|y_n - x_n\| \to 0, \|z_n - y_n\| \to 0.$$

Proof: From the definition of z_n, we obtain $x_n = z_n + b_n(x_n - x_{n-1})$. Obviously, from the closed convexity of the set C, we know z_n ∈ C. Then let $\alpha = z_n$ in the (3.3), we have

$$\|y_n - z_n\|^2 \le \lambda_n\langle F(z_n), z_n - y_n\rangle,$$

also from the definition of λ_n we obtain

$$\|y_n - z_n\| \le \lambda_n\|F(z_n)\| = \frac{a_n}{max\{1, \|F(z_{n-1})\|^2\}}\|F(z_n)\|,$$

when n → ∞, by (3.2) we can get the following inequality

$$\|y_n - z_n\| \le \frac{a_n}{max\{1, \|F(x_n)\|^2\}} \|F(x_n)\| \le a_n \to 0 \tag{3.9}$$

which, together with (3.2) we obtain $\|z_n - y_n\| \to 0(\mathrm{n} \to \infty)$.

Theorem 3.2 If Condition1-Condition3 hold, then the sequence $\{x_n\}$ generated by Algorithm2 converges to a solution x^* of (1.1).

Proof: Firstly, substituting $\alpha = z_n$ into (3.3), we obtain

$$\begin{aligned} \|x_{n+1} - z_n\| &\le \lambda_n \|F(z_n)\| \\ &= \lambda_n \|F(y_n) - F(y) + F(z_n)\| \\ &\le \lambda_n \|F(y_n)\| + \lambda_n \|F(y) - F(z_n)\| \\ &\le a_n + \lambda_n \|F(y) - F(z_n)\| \end{aligned} \tag{3.10}$$

Then from the Condition3 we know there exists constant m > 0 such that

$$\|F(y) - F(z_n)\| \le m, \quad \forall n \ge 0,$$

thus by (3.10) we get

$$\|x_{n+1} - z_n\| \le a_n + m\lambda_n \le a_n + (1 + m), \ \forall n \ge 0 \tag{3.11}$$

By $\{a_n\} \to 0$, then there exists subsequence $\{\lambda_{n_i}\} \subset \{\lambda_n\}$ such that

$$\lambda_{n_i} \left\|F\left(z_{n_i}\right) - F(y_{n_i})\right\| > \rho \left\|z_{n_i} - y_{n_i}\right\|, \ \forall i \ge 1 \tag{3.12}$$

Now we denote

$$\mathrm{K} \coloneqq \{n \in N: \lambda_n \|F(z_n) - F(y_n)\| > \rho \|z_n - y_n\|\} = \{n_i\}_{i=1}^{\infty}.$$

Fix n ∈ K, consider the following two situations:

• If n ∈ K,, from Case1 we have obtained that

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 - (1 - \rho)\|x_{n+1} - y_n\|^2 - (1 - \rho)\|y_n - x_n\|^2. \tag{3.13}$$

• If n ∉ K , there exists i ∈ K such that $n_i = n$, then from (3.12) and $\{a_n\}$ is non-increasing, we can get

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|x_{n+1} - z_n + z_n - x^*\| \\ &\le \|z_n - x^*\| + \|x_{n+1} - z_n\| \\ &\le \|z_n - x^*\| + a_n(m + 1)\delta \\ &\le \|z_n - x^*\| + a_1(m + 1)\delta^i. \end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$\|x_{n+1} - x^*\| \le \|z_n - x^*\| + \xi_n,$$

where

$$\xi_n = \begin{cases} 0, if \ \ n \in K \\ a_1(1 + m)\delta^i, otherwise. \end{cases}$$

Then by (3.2), let n → ∞, we get $\|x_{n+1} - x^*\| \le \|z_n - x^*\| + \xi_n$, owing to δ ∈ (0,1), this implies that

$$\sum_{i=1}^{\infty}\delta_i < \infty, so\ \sum_{i=1}^{\infty}\xi_n < \infty.$$

Combining Lemma3.2 and Lemma2.3, we get

$$\lim_{n\to\infty}\left\|F(z_{n_i}) - F(y_{n_i})\right\| = 0.$$

Now let $\{a_n\} = \{\|x_n - x^*\|\}, \{b_n\} = \{\xi_n\}$, it implies from Lemma2.2 that the sequence $\{\|x_n - x^*\|\}$ is convergent. Also since$\langle a - y_n, z_n - \lambda_n F(y_n) - y_n\rangle \leq 0$, for all α ϵ C, i ϵ N, by (3.12) we obtain

$$\lambda_{n_i}\langle F(z_{n_i}), a - y_{n_i}\rangle \geq \langle a - y_{n_i}, z_{n_i} - y_{n_i}\rangle,$$

i.e.,

$$\begin{aligned}\langle F(z_{n_i}), a - y_{n_i}\rangle &\geq \frac{1}{\lambda_{n_i}}\langle a - y_{n_i}, z_{n_i} - y_{n_i}\rangle \\ &\geq -\frac{1}{\lambda_{n_i}}\|a - y_{n_i}\| \cdot \frac{\lambda_{n_i}}{\rho}\|F(z_{n_i}) - F(y_{n_i})\| \\ &= -\frac{1}{\rho}\|a - y_{n_i}\|\|F(z_{n_i}) - F(y_{n_i})\| \qquad (3.15)\end{aligned}$$

Since the sequence $\{x_n\}$ is bounded and $\{\|x_n - x^*\|\}$ is convergent, so there exists subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to \bar{x}$. Also by the continuity of F and the boundedness of the λ_n, then for ∀α ∈ C, i → ∞, from (3.15) we have

$$\langle F(\bar{x}), \alpha - \bar{x}\rangle \geq 0, \qquad \forall\alpha \in C,$$

hence, $\bar{x} \in C^*$, which implies that

$$\lim_{n\to\infty}\|x_n - \bar{x}\| = \lim_{n\to\infty}\|x_{n_i} - \bar{x}\| = 0.$$

4. CONCLUSIONS

In this paper, we first give an inertial extragradient algorithm for solving variational inequality problem. Under the assumptions of pseudo-monotonicity and non-Lipschitz continuity of F, we verify the global convergence of the algorithm.

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