ON CONSTRUCTION AND APPROXIMATION OF RANDOM SIGNALS

Ioan Golet & Ciprian Hedrea

Dept. of Mathematics, Politehnica University, 300006 Timisoara, Romania.

ioan.golet@mat.upt.ro

ABSTRACT

The signal construction and approximation by sampling at different time moments is one of major target that plays an important role in numerous applications. In many cases, the separation of frequency bands representing periodic or almost periodic behaviors, allows comprehension of the hidden stochastic phenomena involved. In this work the construction and the prediction of random signals based on probabilistic distributions of the random signal at the sampling moments are related to the case of electromagnetic compatibility, where the unintentional generation, propagation and reception of electromagnetic signal energy are analyzed with reference to the unwanted effects (electromagnetic interference) induced by such energy. The paper shows that the probabilistic normed spaces can be an appropriate framework in the study of some issue in the area of electromagnetic compatibility.

Keywords: Electromagnetic compatibility, Approximation of random signal, Probabilistic norm.

1. INTRODUCTION

Almost all naturally occurring phenomena are non-deterministic, that is, they are not entirely predictable. Non-deterministic signals are often called random, probabilistic or stochastic. An ensemble of random variables forms a random signal. A random variable is a mapping from a naturally occurring phenomenon to a vector space with some probability assigned to it. In communication, radar, electrical circuits, computer and telephone networks and many other cases, most signals are random in nature. This is mainly due to the unavoidable omnipresence of electromagnetic noise [4], [6], [9]. The electromagnetic compatibility studies the unintentional generation, propagation and reception of electromagnetic energy with reference to the unwanted effects (electromagnetic interference) induced by such energy. Electromagnetic compatibility pursues two different kinds of issues. The first kind is given by emission issues, which are related to the unwanted generation of electromagnetic energy. The main issues are devoted to the reducing such generation and to avoid the escape of any remaining energies into the external environment. The second kind of issues refer to the correct operation of electrical equipment in the presence of unplanned electromagnetic disturbances. There are different categories of electromagnetic interference according to the source and the characteristics of signals. Continuous interference arises where the source regularly emits a given range of frequencies. In that framework the signal construction and approximation by sampling at different time moments is one of major target that plays an important role in this area. Time series obtained by such measuring processes show either a combination of periodic phenomena with stochastic components or chaotic behavior. Usually, the computing of nonlinear characteristics indicates the real complexity of the system. In many cases, the separation of frequency bands representing periodic or almost periodic behaviors, allows comprehension of the hidden nonlinear or stochastic phenomena involved.

In this work the construction and the prediction of continuous signals are based on probabilistic distributions of the signal at the sampling moments. At a given moment a random signal is assimilated to a random variable on a space with a probability measure. In this paper at each moment a random signal is given by its probability measure, that is, the value of a random signal at the moment is considered a vector of a probabilistic normed space.

2. RANDOM SIGNALS IN PROBABILISTIC NORMED SPACES

We first recall some notions of probabilistic normed spaces. In [12] K. Menger proposed the probabilistic concept of distance by replacing the number by a probabilistic distribution function . This idea led to a large development of probabilistic analysis. Applications to systems having hysteretic, mixture processes, the measuring error were given [15]. In [16] A. N. Šerstnev used K. Menger's idea for systems endowed with an algebraic structure of linear space. So, he initiated the study of probabilistic normed spaces. In [1], [6-7] new classes of probabilistic normed spaces which also include the probabilistic normed spaces defined by A. N. Šerstnev as a special case were studied.

Let denote the set of real numbers, and . The closed unit interval. A mapping is called a distribution function if it is non decreasing, left continuous with inf and sup . denotes
the set of all distribution functions for that \( F(0) = 0 \). Let \( F, G \) be in \( D^\ast \), then we write \( F \leq G \) if \( F(t) \leq G(t) \) for all \( t \in P \). If \( a \in R \), then \( H_a \) will be the element of \( D^\ast \) defined by \( H_a(t) = 0 \) if \( t \leq a \) and \( H_a(t) = 1 \) if \( t > a \). It is obvious that \( H_0 \) \( = F \), for all \( F \in D^\ast \).

A \( t \)-norm \( T \) is a two place function \( T : I \times I \to I \) which is associative, commutative, non decreasing in each place and such that \( T(a, 1) = a \), for all \( a \in [0, 1] \).

A triangle function \( \tau \) is a binary operation on \( D^\ast \) which is commutative, associative, non decreasing in each place and for which \( H_0 \) is the identity, that is, \( \tau(F, H_0) = F \) for every \( F \in D^\ast \). \( T \)-norms and triangle functions have been very important in writing an appropriate probabilistic triangle inequality.

Let \( \varphi \) be a function defined on the real field \( P \) into itself with the following properties: (a) \( \varphi(\cdot t) = \varphi(t) \), for every \( t \in R \); (b) \( \varphi(1) = 1 \); (c) \( \varphi \) is strict increasing and continuous on \( [0, \infty) \), \( \varphi(0) = 0 \) and \( \lim_{\alpha \to \infty} \varphi(\alpha) = \infty \).

**Definition 1.** Let \( L \) be a linear space, \( \tau \) a triangle function and let \( F \) be a mapping from \( L \) into \( D^\ast \). If the following conditions are satisfied:

1. \( F_x = H_0 \) if, and only if, \( x = 0 \);
2. \( F_{a \cdot t} = F_x \left( \frac{t}{\varphi(\alpha)} \right) \), for every \( t > 0 \), \( a \in P \) and \( x \in L \);
3. \( F_{x + y} \geq \tau(F_x, F_y) \), whenever \( x, y \in L \);

the triple \( (L, F, \tau) \) is called a probabilistic \( \varphi \)-normed space (of Šerstnev type).

If (1)-(2) are satisfied and the probabilistic triangle inequality (3) is formulated under a \( t \)-norm \( T \):

4. \( F_{x + t_1 + t_2} \geq T(F(x(t_1)), F(t_2)) \), for all \( x, y \in L \) and \( t_1, t_2 \in R \), then \( (L, F, T) \) is called a Menger \( \varphi \)-normed space.

**Remark 1.**

For \( \varphi(a) = |a|^p \), \( 0 < p \leq 1 \) one obtains a probabilistic \( p \)-normed space, for \( \varphi(a) = |a| \) one obtains an probabilistic normed normed space.

We will use probabilistic framework given by probabilistic normed spaces in study of some issues in related with random signals in electromagnetic compatibility.

Let \( (\Omega, K, P) \) be a complete probability measure space, i.e., the set \( \Omega \) is a non-empty abstract set, \( K \) is a \( \sigma \)-algebra on \( \Omega \) and \( P \) is a complete probability measure on \( K \). Let \( (E, B) \) be a measurable space, where \( (X, || \cdot ||) \) is a separable Banach space and \( B \) is the \( \sigma \)-algebra of the Borel subsets of \( X \).

A mapping \( \xi : \Omega \to X \) is said to be a random variable with values in \( X \) if \( \xi^{-1}(B) \in K \) for all \( B \in B \). Two \( X \)-valued random variables \( x(\omega) \) and \( y(\omega) \) are said to be equivalent if \( x(\omega) \) and \( y(\omega) \) are equal in the probability. Let \( V \) be the set of all class of such \( X \)-valued random variables and let \( F \) be the probabilistic norm on \( V \) defined by

\[
F_x(t) = P\{\omega \in \Omega : ||x(\omega)|| < t\}
\]

It is known that \( (V, F, \tau_{T_m}) \) is a complete probabilistic normed space of Šerstnev type. Furthermore, the \( (\varepsilon, \lambda) \)-topology on \( V \) induced by the probabilistic norm \( F \) is equivalent to the topology of the convergence in probability on \( V \).

Random signals have had a special importance in the probability theory as well as in its applications. Regarding time series as random signals their predictability have increased and random signals have given important new tools in solving economics and engineering problems. In the same time, the properties of a random signals as the coincidence or near coincidence points are useful in analysis of electromagnetic compatibility. These considerations have determined us to approach the study of random signals as functions with values into a probabilistic normed space.

Generally, a mapping \( f \) is said to be a random signal defined on the subset \( A \) of real line with values in a separable Banach space \( X \) if, for every \( t \in A \) the mapping \( f(t, \cdot) : \Omega \to X \) is an \( X \)-valued random variable. Two \( X \)-valued random signals \( f \) and \( g \) are said to be equivalent if \( f(t, \omega) \) and \( g(t, \omega) \) are equal almost surely for every \( t \in A \). Now, let \( f \) be a \( X \)-valued random function defined on \( A \subset R \), then one can defines the mapping \( f \) on \( A \) with values in the random normed space \( (V, F, T_m) \) by \( \forall t \to \tilde{f}(t) \), where \( \tilde{f}(t)(\omega) = f(t, \omega) \).
Conversely, for each function \( \tilde{f} : A \rightarrow (V, F, T_m) \) one can define the X-valued random signal on A by \( f(t, \omega) = \left[ \tilde{f}(t) \right](\omega) \), for every \( t \in A \) and \( \omega \in \Omega \). Furthermore the correspondence \( f \rightarrow \tilde{f} \) is one to one and onto. By this correspondence we can mean by a function with values in a probabilistic normed space a random signal with values in a separable Banach space and in a particular case we identify the two notions.

**Proposition 1.** Let \( f \) be a random signal and let \( (f_n)_{n \in \mathbb{N}} \) be a sequence of random signals defined on a non-empty subset \( A \) of real line with values in a Menger \( \varphi \)-normed spaces \((L, F, T)\). Then:

(a) The random signal \( f \) is continuous in \( t_0 \in A \) if and only if, for every \( \varepsilon > 0 \) and \( \lambda \in (0, 1) \) there is \( \delta(\varepsilon, \lambda) > 0 \) such that for all \( t \in A \) with \( |t - t_0| < \delta(\varepsilon, \lambda) \) we have

\[
F_{f(t)-f(t_0)}(\varepsilon) > 1 - \lambda.
\]

(b) The random signal \( f \) is uniformly continuous on the set \( A \) if and only if, for every \( \varepsilon > 0 \) and \( \lambda \in (0, 1) \) there is \( \delta(\varepsilon, \lambda) > 0 \) such that, for all \( t', t'' \in A \) with property \(|t' - t''| < \delta(\varepsilon, \lambda)\) we have

\[
F_{f(t')-f(t'')} (\varepsilon) > 1 - \lambda.
\]

(c) The sequence \((f_n)_{n \in \mathbb{N}}\) converges on the set \( A \) to the random signal \( f \) if, for every \( \varepsilon > 0 \) and \( \lambda \in (0, 1) \) and \( t \in A \) there is an integer \( N(\varepsilon, \lambda, t) \) such that, for all \( n > N(\varepsilon, \lambda, t) \) we have \( F_{f_n(t)-f(t)}(\varepsilon) > 1 - \lambda \).

(d) The sequence \((f_n)_{n \in \mathbb{N}}\) is uniformly convergent, on the set \( A \) to the random signal \( f \), if, and only if, the condition of the point \((c)\) is satisfied for a \( N(\varepsilon, \lambda, t) \) that doesn’t depend of \( t \).

The above statements are valid because the family \( \{V, (\varepsilon, \lambda) \rightarrow 1 - \lambda, \varepsilon > 0, \lambda \in (0, 1)\} \) is a complete system of neighborhoods for the point \( x \) in the topology generated by the Menger \( \varphi \)-norm \( F \).

**Definition 2.** A sequence \((f_n)_{n \in \mathbb{N}}\) of functions defined on a set \( A \in P \) with values in the Menger \( \varphi \)-normed space \((L, F, T)\) is called a Cauchy sequence if for every \( \varepsilon > 0 \) and \( \lambda \in (0, 1) \) there is an integer \( N(\varepsilon, \lambda) \) such that \( F_{f_n(t)-f_m(t)}(\varepsilon) > 1 - \lambda \), for all \( t \in A \) and \( n, m \geq N(\varepsilon, \lambda) \).

**Theorem 1.** A sequence \((f_n)_{n \in \mathbb{N}}\) of functions defined on the set \( A \) with values in a complete Menger \( \varphi \)-normed space \((L, F, T)\) is uniformly convergent on \( A \) if, and only if, \((f_n)_{n \in \mathbb{N}}\) is a Cauchy sequence on \( A \).

In what follows we analyze the approximation of continuous functions defined on a compact interval with values in Menger \( \varphi \)-normed spaces. First, we give a general approximation theorem based on so-called Borel functions. Using this result we will give a Weierstrass’ type approximation theorem for such functions.

This result can be applied to approximation of random functions with values in a \( \varphi \)-normed space.

As a consequence, a result with regard to the problem of the approximation of continuous functions with values in a linear metric space is obtained.

For each pair of integers \( m, n, 0 \leq m \leq n, n \neq 0 \), let \( g_{mn} \) be the Borel functions defined on the unit interval \([0, 1]\) by:

\[
g(t) = 0, \text{ if } 0 \leq t \leq \frac{m-1}{n} \text{ or } t \geq \frac{m+1}{n}; \quad g(t) = nt-m+1 \text{ if } \frac{m-1}{n} \leq t \leq \frac{m}{n}; \quad g(t) = -nt+m+1 \text{ if } \frac{m}{n} \leq t \leq \frac{m+1}{n}.
\]

In the random normed space \((L, F, T)\) we consider the sequence of random signals \((f_n)_{n \in \mathbb{N}}\) defined by

\[
f_n(t) = \sum_{m=0}^{n} g_{mn}(t) f\left(\frac{m}{n}\right)
\]

**Theorem 2.** If the function \( f \) defined on \([0, 1] \) with values into a Menger \( \varphi \)-normed space \((L, F, T)\) is continuous, then the sequence of functions \((f_n)_{n \in \mathbb{N}}\) defined by \((11)\) is uniformly convergent on \([0, 1]\) to the function \( f \).

**Proof.** Let \( \varepsilon > 0 \) and \( \lambda \in (0, 1) \). Then by continuity of the function \( f \) it follows that there exist \( N(\varepsilon, \lambda) \in \mathbb{N} \) and \( \eta > 0 \) such that \( F_{f(t')-f(t'')}\left(\frac{\varepsilon}{2}\right) > 1 - \frac{\eta}{2} > 1 - \frac{\lambda}{2} \) for all \( t', t'' \in [0, 1] \) with property that \(|t' - t''| < \frac{1}{N(\varepsilon, \lambda)}\).
Let \( n \geq N(\varepsilon, \lambda) \) and \( s \in [0, 1] \), we choose the integer \( k \) such that \( 0 \leq k \leq n \) and \( \frac{k-1}{n} \leq s \leq \frac{k}{n} \). Then, \( s = \frac{k-u}{n} \), where \( 0 \leq u \leq 1 \) and

\[
f_n(s) = \sum_{m=0}^{n} g_{m,n}(s) f\left(\frac{m}{n}\right) = uf\left(\frac{k-1}{n}\right) + (1-u) f\left(\frac{k}{n}\right)
\]

So, we have:

\[
F_{f(s)-f_n(s)}(\varepsilon) = F_{(1-u)f(s)+uf(s)-f_n(s)}(\varepsilon) \geq T \left( F_{(1-u)f\left(\frac{k-1}{n}\right)-\frac{\varepsilon}{2}} \right),
\]

\[
F_{uf\left(\frac{k-1}{n}\right)}\left(\frac{\varepsilon}{2}\right) = T \left( F_{f\left(\frac{k-1}{n}\right)-\frac{\varepsilon}{2\varphi(u)}} \right),
\]

\[
F_{f\left(\frac{k-1}{n}\right)-\frac{\varepsilon}{2\varphi(u)}} \geq T \left( F_{f\left(\frac{k}{n}\right)-\frac{\varepsilon}{2}} F_{f\left(\frac{k-1}{n}\right)-\frac{\varepsilon}{2}} \right) \geq T_m \left( 1 - \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2} > 1 - \eta > 1 - \lambda \right).
\]

The above inequalities imply the conclusion of the theorem.

**Definition 3.** Let \( f : [0, 1] \to (\mathbb{L}, \mathbb{F}, \mathbb{T}) \) be a continuous function. A Bernstein polynomial with values in a Menger \( \varphi \)-normed space \((\mathbb{L}, \mathbb{F}, \mathbb{T})\) associated to the function \( f \) is the function \( B_n : \mathbb{R} \to (\mathbb{L}, \mathbb{F}, \mathbb{T}) \) given by

\[
B_n(t) = \sum_{m=0}^{n} C_n^m f\left(\frac{m}{n}\right) t^m (1-t)^{n-m}
\]

where \( n \in \mathbb{N} \).

**Theorem 3.** Let \((\mathbb{L}, \mathbb{F}, \mathbb{T})\) be a Menger \( \varphi \)-normed space. If \( f : [0, 1] \to (\mathbb{L}, \mathbb{F}, \mathbb{T}) \) is a continuous random signal, then there exists a sequence of Bernstein polynomials with values in \((\mathbb{L}, \mathbb{F}, \mathbb{T})\) associated to the random signal \( f \) which is uniformly convergent on \([0, 1]\) to the random signal \( f \).

The proof of the Theorem 4 is similar to that of Theorem 3 and we omitted it. **COROLLARY 1.**

Let \( f(t, \omega) : I \times \Omega \to X \) be a random signal with values in a separable Banach space \( X \). Then there exists a sequence of random Bernstein polynomials

\[
B_n(t, \omega) = \sum_{m=0}^{n} C_n^m f\left(\frac{m}{n}, \omega\right) t^m (1-t)^{n-m}
\]

which converges in probability to the random signals \( f(t, \omega) \).

We now give some fundamental concepts of differential and integral calculus for random signals with values into probabilistic normed spaces. These will be useful in study of characteristic properties of random signals as being changing spread and amount of energy on intervals.

**Definition 4.** Let \( f \) be a function defined on a subset \( A \) of real line \( \mathbb{R} \) with values in a \( \mathbb{P}N \)-space \((\mathbb{L}, \mathbb{F}, \mathbb{T})\) and let \( t_0 \in I \subseteq A \), where \( I \) is a closed interval. The function \( f \) is said to be derivable in the point \( u \), if there exists an fixed element \( x \) in \( L \) such that, for any \( \varepsilon > 0 \) and \( \lambda \) in \((0,1)\) there exists \( \delta(\varepsilon, \lambda) \) such that

\[
F_{f(t)-f(u)}(\varepsilon) > 1 - \lambda,
\]

for any \( t \in I \setminus \{u\} \) and having the property \( |t-u| < \delta(\varepsilon, \lambda) \). The element \( x \) it is called the derivative of \( f \) in the point \( u \), it is denoted by \( f'(u) \). If \( f \) is derivable in each \( u \) in a open set \( A \) then the function \( f \) is said to be derivable on the set \( A \).
If two random electromagnetic signals possess a coincidence point and derivatives are equal or near equal the persistence of electromagnetic compatibility is almost surely.

**Proposition 2.** If the function \( f \) defined on an open subset \( A \) of real line \( R \) with values into \((L,F,T)\) is derivable in a point \( u \in I \subset A \), then:

(a) the derivative \( f'(u) \) is unique determined;

(b) the random function \( f \) is continuous in the same point \( u \).

**Proposition 3.** A trigonometric polynomial with values into a probabilistic normed space \((L,F,T)\) is a random signal \( P : R \rightarrow L \) defined by:

\[
P(t) = \sum_{k=0}^{n} a_k e^{i\lambda_k t},
\]

where \( a_k \in L, a_n \neq \emptyset, \lambda_k \in R, i = \sqrt{-1}, n \in N \).

**Definition 5.** A continuous random signal \( f \) defined on an open subset of real line \( A \) with values into \((L,F,T)\) is said to be almost periodical if, for every \( \varepsilon > 0 \) and \( \lambda \in (0, 1) \) there exists a trigonometric polynomial \( P(t) = P_{\varepsilon,\lambda}(t) \) such that

\[
F_{f(t) - P(t)}(\varepsilon) > 1 - \lambda,
\]

for every \( t \in R \).

**Theorem 4.** If a random signal \( f : R \rightarrow L \) is almost periodical, then \( f \) is uniformly continuous on \( R \).

**Theorem 5.** (a) If random signals \( f, g : R \rightarrow L \) are almost periodical, then the random signal \( f + g \) is almost periodical on \( R \).

(b) If a sequence of random signals \( (f_n)_{n \in N} \) defined on the set \( R \) with values in a complete probabilistic normed space \((L,F,T)\) is uniformly convergent on \( R \), to the random signal \( f \), then \( f \) is almost periodical on \( R \).

(c) If random signals \( f \) and \( g \) are almost periodical on \( R \) and \( f(R) \) and \( g(R) \) are probabilistic bounded \([15] \), then the product signal \( fg \) is almost periodical \( R \).

**Definition 6.** A continuous random signal \( f \) defined on the real line \( R \) with values into \((L,F,T)\) is said to be Bohr almost periodical if, for every \( \varepsilon > 0 \) and \( \lambda \in (0, 1) \) there exists a positive real number \( l(\varepsilon, \lambda) \) such that for each interval \( I \subset R \) of length \( l(\varepsilon, \lambda) \) there exists \( \tau \in R \) a trigonometric polynomial \( P(t) = P_{\varepsilon,\lambda}(t) \) such that

\[
F_{f(t) - P(t)}(\varepsilon) > 1 - \lambda, \quad \text{for all } t \in R.
\]

The real number \( \tau \) is called almost period of the random signal \( f \).

**Theorem 6.** (a) If a random signal \( f : R \rightarrow L \) is Bohr almost periodical, then \( f \) is uniformly continuous on \( R \).

(b) If random signals \( f, g : R \rightarrow L \) are Bohr almost periodical, then the random signal \( f + g \) is Bohr almost periodical on \( R \).

(c) If a sequence of Bohr almost periodic random signals \( (f_n)_{n \in N} \) defined on the set \( R \) with values in a complete probabilistic normed space \((L,F,T)\) is uniformly convergent on \( R \), to the random signal \( f \), then \( f \) is Bohr almost periodic on \( R \).

Now, let \( f \) be a random signal defined on an interval \( [a,b] \) and let \( d \) be a division of the interval \( [a,b] \) given by \( a = t_0 < t_1 < \ldots < t_n = b \). Let us denote by \( V(d) \) the norm of the division \( d \). Also for an arbitrary system \( u = (u_i) \), where \( u_i \in [t_i, t_{i+1}] \) we consider the Riemann sums

\[
S_{df(u)} = \sum_{i=0}^{n-1} (t_{i+1} - t_i) f(u_i).
\]
Definition 7. A function \( f \) defined on \([a,b]\) with values into a probabilistic normed space \((L,F,T)\) is said to be Riemann integrable on \([a,b]\) if there exist \( v \) in \( L \) such that, for any \( \varepsilon > 0 \) and \( \lambda \in (0,1) \) there exists \( \delta \) in \( L \) and \( \delta(\varepsilon,\lambda) > 0 \) such as whenever \( \delta(d) < \delta(\varepsilon,\lambda) \), then we have \( F_{\sigma(a)},(\varepsilon) > 1 - \lambda \) for any choosing of the system \( u=(u_i) \). The vector \( v \) in \( L \) it is called the Riemann integral of the function \( f \) on the interval \([a,b]\) and it is denoted by
\[
v = \int_a^b f(t) \, dt.
\]

Proposition 4. If the random signal \( f \) defined on an open subset of real line \( A \) with values into a probabilistic normed space \((L,F,T)\) is integrable on \([a,b]\), then \( \int_a^b f(t) \, dt \) is unique determined.

Teorem 7. Let \((L,F,T)\) be a complete probabilistic normed space under a continuous \( t \)-norm \( T \) of Hadžić’s type [8]. If \( f \) is a continuous random signal defined on \([a,b]\) with values in \((L,F,T)\), then \( f \) is Riemann integrable on \([a,b]\).

3. CONCLUSION

Starting from issues yielded in electromagnetic compatibility signals, in this paper we give methods for constructing continual random signals starting from a discrete sampling. We define a derivative and an integral as measures, in probability, of the variation and of the energy for random signals, which will be used in the electromagnetic compatibility and in the signal prediction.

4. REFERENCES

[1]. C. Alsina, B. Schweizer and A. Sklar, On the definition of a probabilistic normed space, Aequationes Mathematicae. 46 (1993), 91-98.