AN INVESTIGATION ON FIBER OPTICAL SOLITON IN MATHEMATICAL PHYSICS AND ITS APPLICATION TO COMMUNICATION ENGINEERING

Md. Haider Ali Biswas, Md. Ashikur Rahman & Tapasi Das
Mathematics Discipline, Khulna University, Khulna-9208, Bangladesh
E-mail: mhabiswas@yahoo.com, ashi_08@hotmail.com, tapasi_ku@yahoo.in

ABSTRACT
Solitons are self-localized wave packets arising from a robust balance between dispersion and nonlinearity. Soliton is the physics of wave, acting upon wave. In mathematics and physics, a soliton is a self-reinforcing solitary wave that maintains its shape while it travels at constant speed. They are a universal phenomenon, exhibiting properties typically associated with particles. Optical soliton in media with quadratic nonlinearity and frequency dispersion are theoretically analyzed over the years. Our aim is to discuss the behavior of soliton solutions to the KdV equation and their interactions and applications are then investigated in the fiber optics solitons theory in communication engineering. In this study optical soliton is studied with illustrated graphical representation.

Key words: Soliton solution, Korteweg de Varies equation, Gaussian white noise, Stochastic KdV equation, Fourier transform.

1. INTRODUCTION
In recent years there have been important and tremendous developments in the study of nonlinear waves and a class of nonlinear wave equations which arise frequently in applications. The wide interest in this field comes from the understanding of special waves called solitons and the associated development of a method of solution to a class of nonlinear wave equations termed the nonlinear Korteweg and de Vries (KdV) equation. A Soliton phenomenon is an attractive field of present day research not only in nonlinear physics and mathematics but also in fiber optics and communication engineering. The soliton phenomenon was first pioneered by John Scott Russel in 1844. The paper by Korteweg and de Vries in 1895 [8] was one of the first theoretical treatment in the soliton solution and thus a very important milestone in the history of the development of soliton theory. The brief discussion of mathematical representation of soliton begins with the Wadati’s paper published in 1983 [16]. Russel L. Herman a famous Mathematician improved soliton theory and found some improved results which represent a final solution of Soliton [6]. Basically Wadati and Herman both used a non-linear third order partial differential equation known as Korteweg de Varies equation, they started from this equation and finally gave mathematical assumption of soliton with graphical representation [14]. One of the active area of applications of solitons is fiber optics. Much experimentation has been done using solitons in fiber optics applications. In 1973, Robin Bullough (see [3] for details) showed that solitons could exist in optical fibers while presenting the first mathematical report of the existence of optical solitons. He also proposed the idea of a soliton-based transmission system to increase performance of optical telecommunications. Now soliton is playing an essential role in interactions with communication engineering. There are varieties of nonlinear equations representing the solitons in the nonlinear domain such as, general equal width wave equation (GEWE), general regularized long wave equation (GRLW), general Korteweg–de Vries equation (GKdV), general improved Korteweg– de Vries equation (GIKdV), and Coupled equal width wave equations (CEWE), which are the important soliton equations. See for examples [1, 2] for more details. Our aim is to investigate some aspects of Korteweg– de Vries equation in soliton physics specially for the case of optical soliton in mathematical physics. We also provide an illustration with some graphical representations.

2. MATHEMATICAL ASSUMPTION OF SOLITON SOLUTION
We consider a partial differential equation of the form,

\[ u_t - 6uu_x + u_{xxx} = 0 \]  \hspace{1cm} (1)

which is called the Korteweg de Varies equation. It is often called non-linear partial differential equation (PDE) of the form,

\[ \frac{\partial \eta}{\partial t} = \frac{3}{2} \sqrt{\eta \frac{\partial \eta}{\partial x}} + \frac{\partial}{\partial x} \left( \frac{2}{3} \frac{\partial \eta}{\partial x} + \frac{\sigma}{3} \frac{\partial^3 \eta}{\partial x^3} \right) \]  \hspace{1cm} (2)

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which is a non-linear shallow water wave equation. Here $\sigma = \frac{h^3}{3} - \frac{\eta u}{\rho g}$, $h$ is channel height, $T$ is surface tension, $g$ is gravitational acceleration and $\rho$ is density.

In order to calculate that in (1), we consider, another PDE known as Kadomtsev [7] equation defined by

$$\frac{3}{4} u_y + w_x = 0 \quad (3)$$

Differentiating (1) partially we get

$$\frac{\partial}{\partial x} (u_x - 6u_x + u_{xxx}) \pm u_{yy} = 0 \quad (4)$$

$$\Rightarrow u_x = u_{xxx} + 3u_{yy} - 6u_x^2 + 6u_x u_{xx}$$

Also differentiating partially and successively we get

$$u_x + \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} (u - 2u^3 - \frac{\partial u}{\partial x}) \right) \right) = 0$$

which after details calculations yields

$$\Rightarrow u_x + u_{xxx} + 6uu_x = \frac{u}{2t}$$

$$\Rightarrow u_x + u_{xxx} + 6uu_x = \xi(t) \quad (5)$$

Here $\xi(t)$ represents a time dependent Gaussian [5] white noise. The stochastic process is called Gaussian white noise if its statistical average is zero i.e.; $\langle \xi(t) \rangle = 0$.

And two covariance functions is given by,

$$\xi(t) \bar{\xi}(T + t) = \sigma^2 \delta(t) \quad (6)$$

For the Fourier transformation of stationary two times covariance function we obtain,

$$F(\omega) = \int dt \langle \xi(t) \bar{\xi}(T + t) \rangle e^{i\omega T} \Rightarrow F(\omega) = \sigma^2 \int dt \delta(t) e^{i\omega T}$$

$$\Rightarrow F(\omega) = \sigma^2 \quad (7)$$

In other words it does not depend upon $\omega$ because there is no co-relation in time. This is why it is called white noise.

Now for simplicity, let us assume a one dimensional stochastic differential equation with additive noise,

$$\frac{dx(t)}{dt} = a(x(t), t) + \eta(t) \quad (8)$$

Here $a(X(t), t)$ is a Langiven (see for example [9]) equation that can be interpreted as a deterministic or average drift term perturbed by a noisy diffusion term $\xi(t)$. For the increase $dx$ during a time step $dt$, we get

$$dx(t) = a(x(t), t) dt + \eta(t) \quad (9)$$

where, $d\eta(t) = \int \eta(t') dt'$

And we assume that

$$d\eta(t^2) = \int dt_t \int dt_{t'} \langle \eta(t) \eta(t') \rangle$$

$$= \int dt_t \int dt_{t'} \sigma^2 \delta(t - t')$$

$$= \sigma^2 dt \quad (10)$$

As long as the intervals $[t, t + dt]$ and $[t', t' + dt']$ which is a true successive step we get

$$\langle d\eta(t) d\eta(t') \rangle = 2\sigma^2 (t - t') \quad (11)$$

If $\delta = \varepsilon$ then in general we write,

$$\langle \xi(t) \xi(t') \rangle = 2\sigma^2 (t - t') \quad (12)$$

For such time dependent noise the stochastic can be transformed into unperturbed KdV equation [8] in the form,
\[ U_T + 6UU_X + U_{XXX} = 0 \] (13)

Let us now introduce the Galilean transformation,
\[ u(x, t) = U(X, T) + \omega(T), \quad X = x + m(t), \quad T = t, \quad m(t) = -\sigma \int_0^t \omega(t') dt' \]

Under the above transformation we have from calculus that the derivatives transform as
\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial X} + \frac{\partial}{\partial T} \frac{\partial}{\partial x} = \frac{\partial}{\partial X} \quad \text{and,} \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial x} \frac{\partial}{\partial T} \frac{\partial}{\partial t} = -\sigma \omega(T) \frac{\partial}{\partial X} + \frac{\partial}{\partial T} \frac{\partial}{\partial t}
\]

Using this transformation we have,
\[
e(t) = u_t + 6uu_x + u_{xxx}
\]
\[
= (U + W)_T - 6WU_X + 6(U + W)U_X + U_{XXX}
\]
\[
= U_T + 6UU_X + U_{XXX} + \omega_T
\]

Let us now define,
\[
\xi = \omega_T
\]
or,
\[
\omega(t) = \int_0^t \xi(t') dt'
\]

which leads to the KdV equation.

We next consider one soliton solution. In such case, let us consider,
\[
U(X, T) = 2\eta \sec^2(\eta(X - 4\eta^2T - X_0))
\]

Then the above transformation leads directly to an exact solution of stochastic KdV equation,
\[
u(x, t) = 2\eta^2 \text{sech}^2 \left(\eta \left(x - 4\eta^2t - x_0 - 6\int_0^t \omega(t') dt'\right)\right) + \omega(t)
\]

\[\Rightarrow \langle u(x, t) \rangle = 2\eta^2 \text{Sech}^2 (\eta(\eta(-4\eta^2t - x_0 - 6\int_0^t \omega(t') dt')) \]

Formally we can write
\[
\text{sech}^2z = \frac{4}{(e^z + e^{-z})^2} = \frac{4e^{-4\eta t}}{(1 + e^{-4\eta t})^2}
\]

Wadati then presented by computing
\[
\langle u(x, t) \rangle = 8\eta^2 \sum_{n=1}^{\infty} (-1)^{n+1} \left\{ \exp[2m(\eta(-4\eta^2t - x_0 - 6\int_0^t \omega(t') dt'))] \right\}
\]
\[
= -2 \frac{d}{dz} \frac{1}{1 + e^{z}} = -2 \frac{d}{dz} \left( \sum_{n=1}^{\infty} (-e^{-nz})^n \right) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} ne^{-2nz}
\]

In order to complete this composition some following useful relations are needed.
\[
\langle \omega(t) \rangle = 0
\]
\[
\langle \omega(t_1) \omega(t_2) \rangle = 2\epsilon \min(t_1, t_2)
\]
\[
\langle \exp(\omega(t)) \rangle = \exp \left(\frac{1}{2} e^2 \langle \omega^2(t) \rangle \right)
\]

Applying this we get
\[
\exp \left( \pm 12m \int_0^t \omega(t') dt' \right) = \exp \left( 72n^2 \eta^2 \int_0^t \langle \omega(t_1) \omega(t_2) dt_1 dt_2 \rangle \right)
\]
\[
= \exp(48n^2 \eta^2 t^3)
\]

This leads to the following form,
\[ \langle u(x, t) \rangle = 8\eta^2 \sum_{n=1}^{\infty} (-1)^n n e^{na+nb^2} \]  

where \( a = 2\eta(x-x_0-2\eta^2 t), \ b = 48\eta^2 e^t \)  

In principle this should be sufficient but we go on to find the expression that analytically give an expression to this result. 

Now the differentiation of the series with respect to \( a \) and \( b \) respectively leads to the partial differential equation  

\[ w_b = w_{aa} \text{ for } w(a, b) = \langle u(x, t) \rangle. \]

Furthermore, we have that  

\[ w(a, 0) = 2\eta^2 \text{Sech}^2 \frac{a}{2}. \]

This is an initial value problem for the heat or diffusion equation on the real line. To solve it we use Fourier transformation. The Fourier transform is defined by,  

\[ \mathcal{F}\{w(a, b)\} = \int_{-\infty}^{\infty} \mathcal{F}\{w(a, b)e^{-iak}\} da \]

And the inverse transform,  

\[ w(a, b) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}^{-1}\{w(k, b)e^{iak}\} dk \]

The heat equation leads to the simple initial value problem  

\[ w_b = -k^2 \bar{w} \]

where  

\[ \bar{w}(k, t) = 2\eta^2 \int \text{Sech}^2 \frac{a}{2} e^{-ak} da \]

\[ = 8\eta^2 \frac{\pi k}{\sinh \pi k} \]

Therefore,  

\[ \bar{w}(k, b) = 8\eta^2 \frac{\pi k}{\sinh \pi k} e^{-bk^2} \]  

(19)

And the solution is thus found out from inverse Fourier transform as  

\[ u(x, t) = \frac{4\eta^2}{\pi} \int_{-\infty}^{\infty} \frac{\pi k}{\sinh \pi k} e^{-bk^2} dk \]  

(20)

Wadati indicated that this was simply done using the convolution theorem. Namely we note that  

\[ \tilde{w}(k, b) = \tilde{f}(k)\tilde{g}(k, b) \]  

for  

\[ \tilde{f}(k) = 8\eta^2 \frac{\pi k}{\sinh \pi k} \]  

and \( \tilde{g}(k, b) = e^{-bk^2} \)

The inverse transform for these are given by  

\[ f(a) = 2\eta^2 \text{sech}^2 \frac{a}{2} \]  

and, \( g(a, b) = \frac{1}{\sqrt{4\pi b}} e^{-a^2/4b} \)

The last expression is just the statement for Fourier transformation of a Gaussian. Now from the Gaussian Convolving of the functions, we have  

\[ \langle u(x, t) \rangle = \omega(a, b) = (f * g)(a) = \int_{-\infty}^{\infty} f(s)g(a-s)ds = \int_{-\infty}^{\infty} 2\eta^2 \text{sech}^2 \frac{s}{2} \left( \frac{1}{\sqrt{4\pi b}} e^{-\frac{(a-s)^2}{4b}} \right) ds \]

\[ = \frac{\eta^2}{\sqrt{\pi b}} \int_{-\infty}^{\infty} e^{-\frac{(a-s)^2}{4b} \text{sech}^2 \frac{s}{2}} ds \]  

(21)

This is the exact solution to which we can compare any simulation results. The graph of simulation solution is given next in Fig. 1.
Fig. 1 The solution generated by doing a simulation of the stochastic KdV equation.

Most of the focuses of any simulations are with respect to the asymptotic results that Wadati derived from the above solution. Namely, for small times \( b = 48\eta^2 \varepsilon t^2 < 1 \) it is a simple matter to show that,

\[
\langle u(x, t) \rangle = 2\eta^2 \sum_{n=0}^{\infty} \frac{b^n}{n!} \frac{\partial^{2n}}{\partial a^{2n}} \text{sech}^2 \frac{a}{2} 
\]

Fig. 2 indicates the numerically illustrated exact solution which is given below.

Fig. 2 The graph of exact solution using equation (22).

Now we present here the comparison between Fig. 1 and Fig. 2 in the following graph (see Fig. 3).
For large times \( b = 48\eta^2\epsilon t^2 > 1 \) we can show that

\[
\langle u(x, t) \rangle = \frac{4\eta^2}{\sqrt{\pi}} \left( 1 + \sum_{n=1}^{\infty} \frac{(2^{2n} - 2)bB_n\pi^{2n}}{(2n)!} \frac{\partial^n}{\partial b^n} \right) e^{-x^2/4b}
\]

(23)

Graph in large time solution is presented in Fig. 4.

Most focus on the case when \( t \to \infty \) result that (see the Fig. 5)

\[
\langle u(x, t) \rangle \approx \frac{\eta}{\sqrt{3\pi}c} \frac{1}{t^{3/2}} \exp \left( -\frac{(x - x_0 - 4\eta^2 t)^2}{48\epsilon t^3} \right)
\]

(24)
Fig. 5: The solution for large times based upon Equation (24).

Fig. 6: The amplitude of the solutions in the Fig. 4 and Fig. 5.

Thus an extensive graphical illustrations are presented as the final resultant we get from the above solutions as well as simulations.

3. A PARTICULAR PROBLEM & SOLUTION

There are several problems with Wadati’s derivation. These also appear elsewhere in the literature references to Wadati’s paper [16]. Here we discuss a particular problem.

First, we note that the series expansion for the \( \sec h^2 z \) is not quite right. We should instead have derived it as follows (for \( z \neq 0 \))

\[
\text{Sech}^2 z = \frac{4}{(e^z + e^{-z})^2} = \frac{4e^{-2|z|}}{(1 + e^{-2|z|})^2} = 2\text{sgn} \frac{d}{dz} \left( \sum_{n=0}^{\infty} (-e^{-2|z|})^n \right)
\]
\[ \text{Sech}^2 z = 2 \sum_{n=0}^{\infty} (-1)^n e^{-2|z|^2} \]  
(25)

This accounts for the convergence of the geometric series used in the derivation. Namely, in the original derivation, one should have noted that \( |e^{-2z^2}| < 1 \); or \( z < 0 \).

This new derivation accounts for the case \( z > 0 \). Konotop and Vazquez appeared to have used this in their review of Wadati’s derivation [7]. They presented the infinite series result as,

\[ \langle u(x, t) \rangle = 8\eta^2 \sum_{n=0}^{\infty} (-1)^n e^{-np|z|^2} \]  
(26)

There also appeared to be a problem with the derivation of the average, where Wadati should actually have computed

\[ \langle u(x, t) \rangle = 8\eta^2 \sum_{n=0}^{\infty} (-1)^n \left\{ \exp \left( -2n\eta |x - 4\eta^2 t - x_0 - 6 \int_0^t \omega(t') dt' \right) \right\} \]  
(27)

One could get around this problem by computing the average for space-time regions where \( x - 4\eta^2 t - x_0 - 6 \int_0^t \omega(t') dt' \) is definitely of one sign. Another approach would instead be to be directly expanded as

\[ u(x, t) = 2\eta^2 \text{Sech}^2 \left\{ \eta (x - 4\eta^2 t - x_0 - 4\eta^2 t) - 6 \int_0^t \omega(t') dt' \right\} = 2\eta^2 \text{sech}^2 (\theta + \sigma) \]  
(28)

In case about \( \sigma = 0 \) for \( \sigma = -6 \int_0^t \omega(t') dt' \)

We have that

\[ 2\eta^2 \text{sech}^2 (\theta + \sigma) = 2\eta^2 \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} \frac{\partial^n}{\partial \theta^n} \text{sech}^2 \theta \]  
(29)

The average can now be computed as

\[ \langle u(x, t) \rangle = 2\eta^2 \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} \frac{\partial^n}{\partial \theta^n} \text{sech}^2 \theta \]  
(30)

provided that we can compute

\[ \langle \sigma^n \rangle = \left\langle \left( -6 \int_0^t \omega(t') dt' \right)^n \right\rangle \]  
(31)

Herman showed that such averages can be computed based upon the nature of the Gaussian noise as,

\[ \langle \sigma^n \rangle = \begin{cases} 0, n = \text{odd} \\ (2l-1)! \langle \sigma^2 \rangle^l, n = 2l, \text{even} \end{cases} \]

Thus, we just need to compute \( \langle \sigma^2 \rangle \).

To complete the result how this is done, we have

\[ \langle \sigma^2 \rangle = \left\langle 36\eta^2 \int_0^t \omega(t_1) dt_1 \int_0^t \omega(t_2) dt_2 \right\rangle = 72\eta^2 \int_0^t \min(t_1, t_2) dt_1 dt_2 \]

\[ = 72\eta^2 \int_0^t \min(t_1, t_2) dt_1 + \int_0^t \min(t_1, t_2) dt_1 \left\{ \frac{1}{2} + t_1(t-t_1) \right\} dt_2 = 24\eta^2 t^3 \]

Intersecting the result in equation (30) yields.
\[ \langle u(x, t) \rangle = 2\eta^2 \sum_{n=0}^{\infty} \left( \frac{\sigma}{n!} \right)^n \partial^n \sech^2 \theta = 2\eta^2 \sum_{n=0}^{\infty} \left( \frac{\sigma}{n!} \right)^n \left( 21 \right) \partial^{21} \sech^2 \theta \\
\langle u(x, t) \rangle = 2\eta^2 \sum_{n=0}^{\infty} \left( \frac{12\eta^2 t^4}{l!} \right)^n \partial^{21} \sech^2 \theta \\
\langle u(x, t) \rangle = 2\eta^2 \sum_{n=0}^{\infty} \left( \frac{48\eta^2 t^4}{l!} \right)^n \partial^{21} \sech^2 \theta \]

In order to see the agreement with Wadati's result for small \( b = 48\eta^3 t^3 \), we need to set \( \theta = \frac{a}{2} \). Noting that \( \partial^{21} \theta = \frac{2a^{21}}{\eta} \), we obtain

\[ \langle u(x, t) \rangle = 2\eta^2 \sum_{n=0}^{\infty} \left( \frac{48\eta^2 t^4}{l!} \right)^n \partial^{21} \sech^2 \theta \frac{a}{2} = 2\eta^2 \sum_{n=0}^{\infty} \left( \frac{b}{l!} \partial^{21} \sech^2 \theta \frac{a}{2} \right) \]

We further note that this solution again satisfies the heat equation and that for \( b = 0 \) this solution reduces to the soliton initial condition. Thus, we have seemingly bypassed any problem with the computing the average with an absolute value. However, this series is divergent for \( b > 1 \).

4. CONCLUSION

Soliton theory has been a challenging area of research over the years, specially its applications in the diverse fields of science and engineering such as nonlinear analysis, water waves, relativistic and quantum field theory as well as electrical and communication engineering have made this theory more attractive. In this study the soliton physics and some of its large scale applications are studied with simulations and its mathematical derivation are shown by using the Korteweg de Varies equation i.e; partial differential equation are used to compare the result of the solution which is generated by doing a simulation of the stochastic KdV equation and the exact solution is generated by using simulation equation as well and the comparison between them is shown graphically. Also the solution in small and large time base is represented graphically.

5. REFERENCES