BOUNDS ANALYSIS OF SINGULAR VALUES FOR REAL SYMMETRIC INTERVAL MATRICES

Fang Su [1,*], Zhan Xu[2], Qiaoli Dong[3] & Housen Li[1]

[1] Department of Mathematics & Systems Science, National University of Defense Technology, Changsha, China
[2] School of Electronic Science & Engineering, National University of Defense Technology, Changsha, China
[3]College of Science, Civil Aviation University of China, Tianjin 300300, China.

ABSTRACT

In this paper, we present a new method for calculating the singular values bounds of an interval matrix. we regard singular values as the the largest eigenvalues of the Jordan-Wielandt matrix. Using the property of eigenvalue bound for interval matrix, we give a method about singular values bounds. This method can analyse stability of systems in control fields extensively. An numerical example illustrating the applicability and effectiveness of the new method is also provided.

Keywords: *Interval matrix; Singular values bounds; Real symmetric matrices; Wely's theorem;*

1. INTRODUCTION

Many real-life problems suffer from diverse uncertainties, as a result of inaccuracy of measurements, errors in manufacture, etc. Therefore, the concept of uncertainty is becoming more and more important. Probability theory is the traditional approach to handling uncertainty. This approach requires sufficient statistical data to justify the assumed statistical distributions. Analysis agree that, given sufficient statistical data, the probability theory describes the stochastic uncertainty well. However, probabilistic modeling cannot handle situations with incomplete or little information on which to evaluate a probability, or when that information is nonspecific, ambiguous, or conflicting. In the mid sixties, the interval analysis was proposed [1]. It turns out to be a very powerful technique to study the variations of a system and to understand its uncertainty. One of the most important properties of this approach is the fact that it is possible to certify the results of all the states of a system.

The problem of computing the singular value bounds of interval matrices has been studied since the nineties. Deif's method [2] produces exact singular value sets, but only under some assumption that are generally difficult to verify. Ahn & Chen [3] presented a method for calculating the largest possible singular value. It is a slight modification of [4] and time complexity is exponential (2^{m+n-1} iterations). They also proposed a lower bound for the smallest possible singular value by means of interval matrix inversion. To get an outer approximation of the singular value set of A, we can apply the eigenvalue bound methods of the symmetric interval matrix

$$\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}.$$

The problem of computing the eigenvalue bounds of interval matrices has many effect results. Deif [5] firstly considered the interval eigenvalue problem. Qiu [6] dealed with the standard interval eigenvalue problem using the vertex solution theorem and the parameter decomposition solution theorem. Zhan [7] presented the range of the smallest and largest eigenvalues of real symmetric interval matrices. Leng [8] obtained the eigenvalue bounds of the original interval eigenvalue problem based on the matrix perturbation. This method is very simple and unconditional but the bounds are not very sharp. Leng [9] presented a new method with two algorithms for computing bounds to real eigenvalues of real-interval matrices.

This paper is structured as follows. In Section 2, we provide the mainresults of singular value bound estimate for symmetric interval matrix and the proof. Then, a numerical example for demonstration effect of our method is given in Section 3.

2. BOUND OF INTERVAL SINGULAR VALUE

The singular values of A are identical with the largest eigenvalues of the Jordan-Wielandt matrix, i.e

$$\sigma_i = \sqrt{\lambda_i(A^T A)}, \qquad i = 1, 2, \cdots, n \tag{1}$$

here $A \in A^I$, interval matrix A^I is defined as

$$A^I = [\underline{A}, \overline{A}] = \{A \in R^{n\times n}; \underline{A} \le A \le \overline{A}\} \tag{2}$$

where $\underline{A}, \overline{A} \in R^{n\times n}, \underline{A} \le \overline{A}$, are given symmetric matrices. It is important to note that not every matrix in A^I is symmetric. Here we only consider the symmetric matrices, and the nonsymmetric parts are considered in our future paper. By

$$A^c = \frac{\overline{A}+\underline{A}}{2} \qquad for\ \ a_{ij}^c = \frac{\overline{a_{ij}}+\underline{a_{ij}}}{2}$$

$$\Delta A = \frac{\overline{A}-\underline{A}}{2} \qquad for\ \ \Delta a_{ij} = \frac{\overline{a_{ij}}-\underline{a_{ij}}}{2}$$

we denote the midpoint and the radius of A^I respectively.

σ is the singular value of the uncertain-but-bounded matrix A. For a given real symmetric interval matrix A^I, find an singular value interval σ^I defined by

$$\sigma^I = [\underline{\sigma}, \overline{\sigma}] = (\sigma_i^I), \qquad \sigma_i^I = [\underline{\sigma}_i, \overline{\sigma}_i]$$

such that it encloses all possible singular values σ, and also it should be as small as possible.

Since the singular values are now not only the points but also the intervals. We will give some results for bounds estimate of singular values, at first we review some knowledge about the Weyl theorem [10].

Theorem 2.1 (Weyl Theorem) Let $A, B \in R^{n\times n}$ be symmetric matrices with eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ and $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$, then

$$|\lambda_i - \mu_i| \le \|A-B\|_2, \qquad (i = 1, 2, \cdots, n) \tag{3}$$

Now we will deduce the following important conclusion according Weyl theorem.

Theorem 2.2 Let $A, B \in R^{n\times n}$ be symmetric matrices with singular value $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n$ and $\tau_1 \ge \tau_2 \ge \cdots \ge \tau_n$, respectively. Then one has

$$|\tau_i - \sigma_i| \le \|A-B\|_2, \qquad (i = 1, 2, \cdots, n) \tag{4}$$

Proof. Let $B - A = E$, then

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} - \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & E \\ E^T & 0 \end{pmatrix}$$

Let the singular values of E are $\varepsilon_1 \ge \cdots \ge \varepsilon_n \ge 0$, the eigenvalues of the three real symmetric matrices $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & E \\ E^T & 0 \end{pmatrix}$ are

$$\tau_1 \ge \cdots \ge \tau_n \ge -\tau_n \ge \cdots \ge -\tau_1$$

$$\sigma_1 \ge \cdots \ge \sigma_n \ge -\sigma_n \ge \cdots \ge -\sigma_1$$

and

$$\varepsilon_1 \ge \cdots \ge \varepsilon_n \ge -\varepsilon_n \ge \cdots \ge -\varepsilon_1$$

Let singular value decomposition for A is $A = U\Sigma V^T$, here U and V are unitary matrices, $\Sigma = diag(\sigma_1, \cdots, \sigma_n)$, it is not difficulty to obtain

$$\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} = \widehat{U} \begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix} \widehat{V}^T$$

Where

$$\widehat{U} = \begin{pmatrix} \frac{1}{\sqrt{2}}U & \frac{1}{\sqrt{2}}U \\ \frac{1}{\sqrt{2}}V & -\frac{1}{\sqrt{2}}V \end{pmatrix}$$

According Weyl theorem, we have

$$|\tau_i - \sigma_i| \le \left\| \begin{pmatrix} 0 & B-A \\ (B-A)^T & 0 \end{pmatrix} \right\|_2, \qquad i = 1, 2, \cdots, n \tag{5}$$

Whereas

$$\left\| \begin{pmatrix} 0 & E \\ E^T & 0 \end{pmatrix} \right\|_2 = \varepsilon_1 = \|E\|_2 \tag{6}$$

And

$$\left\| \begin{pmatrix} 0 & B-A \\ (B-A)^T & 0 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} 0 & E \\ E^T & 0 \end{pmatrix} \right\|_2, \quad \|A-B\|_2 = \|E\|_2 \tag{7}$$

Combining (5), (6) and (7), then

$$|\tau_i - \sigma_i| \le \|A-B\|_2, \quad (i = 1, 2, \cdots, n).$$

The above theorem demonstrates that singular value has good stability property, this is why singular value decomposition has wide applications. Now we will give bounds estimate theorem for singular values.

Theorem 2.3 Let $A \in R^{n\times n}$ be symmetric matrices in A^I , $A^c, \Delta A$ are the midpoint and the radius of A^I respectively. Let the singular values of A are $\tau_1 \geq \cdots \geq \tau_n$, the singular values of A^c are $\sigma_1 \geq \cdots \geq \sigma_n$ and the spectrum radius of ΔA is ρ, then

$$|\tau_i - \sigma_i| \leq \rho, \ (i = 1, 2, \cdots, n). \tag{8}$$

Proof. From the definition of the interval matrix, $A \in A^{\mathrm{I}}$ can be expressed as

$$\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & A^c \\ A^{cT} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \delta A \\ \delta A^T & 0 \end{pmatrix}$$

here $\delta A \in [-\Delta A, \Delta A]$. From Weyl theorem, we can obtain

$$|\tau_i - \sigma_i| \leq \left\| \begin{pmatrix} 0 & \delta A \\ \delta A^T & 0 \end{pmatrix} \right\|_2, (i = 1, 2, ..., n) \tag{9}$$

We know
$$\left\| \begin{pmatrix} 0 & \delta A \\ \delta A^T & 0 \end{pmatrix} \right\|_2 = \|\delta A\|_2 = \rho(\delta A) \tag{10}$$

where $\rho\left(\delta A\right)$ is the spectrum radius of δA. Because $|\delta A| \leq |\Delta A|$, so we have

$$\rho(\delta A) \leq \rho(\Delta A) = \rho \tag{11}$$

Combining (9), (10) and (11), then

$$|\tau_i - \sigma_i| \leq \rho, \ (i = 1, 2, \cdots, n).$$

From Theorem 2.3, we can obtain

$$\sigma_i - \rho \leq \tau_i \leq \sigma_i + \rho, \ (i = 1, 2, \cdots, n)$$

Take $\overline{\tau}_i = \sigma_i + \rho$, $\underline{\tau}_i = \sigma_i + \rho, \ (i = 1, 2, \cdots, n)$. We have obtained the singular values bounds for real symmetric interval matrices.

3. NUMERICAL RESULTS

We consider a spring-mass system with five degrees of freedom as shown in Figure 1. Masses are denoted by m_1, m_2, m_3, m_4, m_5 and springs are denoted by k_1, k_2, k_3, k_4, k_5 and k_6 , respectively.

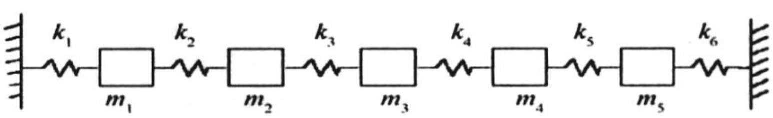

Figure 1 spring-mass system

Let masses of the spring are the unit masses, the stiffness parameters are as follows:

$$[k_1] = [2000, 2020]\, N/m \quad [k_2] = [1800, 1830]\, N/m \quad [k_3] = [1600, 1630]\, N/m$$

$$[k_4] = [1400, 1420]\, N/m \quad [k_5] = [1200, 1230]\, N/m \quad [k_6] = [1000, 1030]\, N/m$$

According to the relationship of the stiffness matrix elements and stiffness parameters, we can obtain stiffness matrix interval as follows:

$$K^I = \begin{bmatrix} [3800, 3850] & -[1800, 1830] & 0 & 0 & 0 \\ -[1800, 1830] & [3400, 3460] & -[1600, 1630] & 0 & 0 \\ 0 & -[1600, 1630] & [3000, 3050] & -[1400, 1420] & 0 \\ 0 & 0 & -[1400, 1420] & [2600, 1650] & -[1200, 1230] \\ 0 & 0 & 0 & -[1200, 1230] & [2200, 2260] \end{bmatrix}$$

The nominal stiffness matrix K^c and the deviation radius matrix of the stiffness matrix are given respectively by

$$K^c = \begin{bmatrix} 3825 & -1815 & 0 & 0 & 0 \\ -1815 & 3430 & -1615 & 0 & 0 \\ 0 & -1615 & 3025 & -1410 & 0 \\ 0 & 0 & -1410 & 2625 & -1215 \\ 0 & 0 & 0 & -1215 & 2230 \end{bmatrix}$$

and

$$\Delta K = \begin{bmatrix} 25 & 15 & 0 & 0 & 0 \\ 15 & 30 & 15 & 0 & 0 \\ 0 & 15 & 25 & 10 & 0 \\ 0 & 0 & 10 & 25 & 15 \\ 0 & 0 & 0 & 15 & 30 \end{bmatrix}$$

The spetrum radius of ΔK is

$$\rho(\Delta K) = 50.6683$$

Let the lower and upper bounds of singular value by using Theorem 2.3 be denoted by $\underline{\sigma}_i$ and $\overline{\sigma}_i$, for i=1,2,3,4,5, respectively. The results are summarized in Table I. For every matrix in K^I, One can test that the singular value must be in the interval [$\underline{\sigma}_i$, $\overline{\sigma}_i$]. It shows that the present method can produce the correct singular value bounds.

Table 1. Lower and upper bounds of singular value

	$\underline{\sigma}_i$	$\overline{\sigma}_i$
σ_1^I	337.4255	438.7621
σ_2^I	1397.2881	1498.6247
σ_3^I	2839.5425	2940.8791
σ_4^I	4325.5137	4426.8503
σ_5^I	5981.8885	6083.2251

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